Al-Qadisiyah Journal of Pure Science Vol. (31) Issue (Special) (2026) pp. 142– 150

Al-Qadisiyah Journal of Pure Science

ISSN(Printed): 1997-2490 ISSN(Online): 2411-3514

DOI: 10.29350/jops

Al-Qadisiyah Journal of Pure Science

On Semi-Strong Metacompact J-spaces, Semi-Weak Metacompact J-spaces and Weak Metacompact J-spaces

Hasan S. Saeed*[1], Raad Aziz Hussain Al-Abdulla[2]

[1]Al-Qadisiyah University, College of Sciences, Department of Mathematics, Al-Qadisiyah/ Iraq

[2]Al-Qadisiyah University, College of Sciences, Department of Mathematics, Al-Qadisiyah/ Iraq

*Corresponding author: sci.math.phd.25.2@qu.edu.iq

Received: 25/03/2026; Revised: 21/04/2026; Accepted: 26/04/2026; Published Online: 26/04/2026.

Abstract

The present work introduces three new spaces: a semi-strong metacompact J-space, a semi-weak metacompact J-space, and a weak metacompact J-space. These spaces are variant of metacompact J-spaces and strong metacompact J-spaces. These spaces represent a generalization of the concepts of metacompact spaces and J-spaces introduced by E. Michael. This paper presents a generalization of J-spaces by weakening certain topological conditions to include a broader class of topological spaces. We investigate the fundamental properties of this new class and establish relationships with existing topological spaces. Several results and properties are obtained that contribute to a better understanding of J-spaces and their generalizations. In the second section of this paper, we review some definitions and theories essential to our work. Section three introduces the basic definitions of spaces semi-strong metacompact J-spaces, semi-weak metacompact J-spaces and weak metacompact J-spaces, and studies their fundamental properties, examples to illustrate some of these properties, and examines of the relationship between these spaces and metacompact J-spaces and strong metacompact J-spaces. Finally, the fourth section examines the relationship between (semi-strong metacompact J-spaces, semi-weak metacompact J-spaces and weak metacompact J-spaces) and (semi-strong J-spaces, semi-weak J-spaces and weak J-spaces), respectively, where the condition of countably compact is necessary for the equivalence of these spaces.

Keywords: MJ-space, strong MJ-space, semi-strong MJ-space, semi-weak MJ-space, weak MJ-space.

1. Introduction

Michael was the first to introduce the concept of a J-spaces in 2000 [1], where what Michael introduced was a generalization of the Jordan curve theory, which is among the classic theories in mathematics [2].

Michael introduced two main spaces, which are, (X,τ) is said to be J-space, when for each $\{K_1,K_2\}$ is a cover of (X,τ) by closed sets such that $K_1 \cap K_2$ is compact, then either K_1 or K_2 is compact. (X,τ) is said to be strong J-space, when for each $L_1 \subset X$ compact, be included in a compact $L_2 \subset X$ so that $X \setminus L_2$ is connected. Michael also presented three other spaces closely related to the J-space and strong J-space, which are. The space (X,τ) is defined as semi-strong J-space, if it is achieved for each $L_1 \subset X$ compact there is compact subset L_2 of (X,τ), with the property that $L_1 \subset L_2$ and there is connected subset $C \subset X$ with $C \subset X \setminus L_1$ and $C \cup L_2 = X$. The space (X,τ) is defined as semi-weak J-space, if it is achieved for every K_1 and K_2 are closed subsets of (X,τ) such that $K_1 \cap K_2 = \emptyset$ and $\partial K_1, \partial K_2$ are compact (where ∂K_1 is a boundary of K_1), then K_1 or K_2 is compact. The space (X,τ) is called weak J-space, if it is achieved for each $\{K_1,K_2,K_3\}$ is a closed covering of (X,τ) such that K_3 compact and $K_1 \cap K_2 = \emptyset$, then K_1 or K_2 is compact, see [1].

We introduced metacompact J-spaces and strong metacompact J-spaces in a forthcoming research paper titled" Some New Generalized Types of J-spaces and Metacompact spaces", where they are defined as follows, the space $(X,\tau)$is called MJ-space (or metacompact J-spaces), if is achieved for each cove $\{K_1,K_2\}$ of (X,τ) by closed sets such that $K_1 \cap K_2$ metacompact, so either K_1 or K_2 is metacompact. The space (X,τ) is called strong MJ-space (or strong metacompact J-spaces), if (X,τ) is achieved for all $L_1 \subset X$ metacompact be included in $L_2 \subset X$ is a closed metacompact such that $X \setminus L_2$ is connected, see [3].

Over the past few years, several authors have presented generalizations of J-spaces, exploring the relationship between J-spaces and these generalized spaces and the conditions under which they coincide or differ. For example, in 2007, Y.-Z. Gao presented the first generalization of J-spaces by using the concept of Lindelöf spaces, demonstrating that these spaces are distinct and exhibit interesting behaviors [4].

In 2023, S. S. Mthethwa and A. Taherifar presented a study of J-spaces and other related types of spaces by using the concept of relatively connected subsets [5].

In 2024, S. S. Mthethwa and A. Taherifar presented algebraic descriptions of J-spaces and C-normal spaces, introducing a new class of spaces known as JC-spaces by using the concept of a Z-connected ideal in $C(X)$ [6].

In 2025, Siyabonga Dubazana and Simo S. Mthethwa presented a generalized topological structure inspired by J-spaces and their pointfree analog, J-frames. The aim is to extend the compactness and separation behavior characteristic of J-spaces to a wider class while preserving their basic properties [7]. In 2025, the same researchers also introduced the concept of JC-frames, which is a generalization of the J-frames concept by replacing the compact condition with the connected condition [8].

2. Preliminaries

In this section we will mention some important definitions, results and examples that will play an important role in proving our results.

Definition 2.1:[9] The space (X, τ) is metacompact, provided that for each U open cover of (X, τ), U has a point finite open refinement.

Remark 2.2:[10] For all compact topological space (X, τ) is metacompact.

Example 2.3:[11] The usual topology on the real numbers set $\mathbb{R}$ is metacompact space but it is not compact.

Definition 2.4:[12] The topological space (X, τ) is called countably compact, if it holds that for each countable open cover of (X, τ) contains a finite subcover.

Definition 2.5:[13] The topological space (X, τ) is called locally compact, if it holds that each

point of X has compact neighbourhood.

Definition 2.6:[14] The topological space (X, τ) is called a Hausdorff space, if it holds that each distinct $a, b \in X$ there exist disjoint open sets U_1, U_1 such that $a \in U_1$ and $b \in U_2$

Theorem 2.7:[15] The metacompact space (X, τ), if it is countably compact, then it is compact.

Theorem 2.8:[16] If the D is a closed subset of metacompact space (X, τ), consequently D is metacompact.

Theorem 2.9:[17] The disjoint union of metacompact spaces is metacompact.

Proposition 2.10:[3] Suppose that (X, τ) is a strong MJ-space, the (X, τ) is an MJ-space.

Remark 2.11:[3] In general, the converse of Proposition (2.10) cannot be true.

Proposition 2.12:[3] Every metacompact space (X, τ) is a strong MJ-space.

Theorem 2.13:[3] Assume that (X, τ) is a topological space, then the following are equivalent.

1. (X, τ) is an MJ-space.

2. If $K \subset X$, such that the boundary of K is metacompact, then $\overline{K}$ or $\overline{(X \setminus K)}$ is metacompact.

3. If K_1 and K_2 are closed subset of Xsuch that $K_1 \cap K_2 = \emptyset$, and if $\partial(K_1)$ or $\partial(K_2)$ is metacompact, then K_1 or K_2 is metacompact.

3. Semi-Strong *MJ*-spaces, Semi- Weak *MJ*- spaces and Weak *MJ*- spaces

In this section we present the basic definitions of semi-strong metacompact J-space, semi-weak metacompact J-space and weak metacompact J-space, as well as some results and examples.

Definition 3.1: The space (X,τ) is defined as semi-strong metacompact J-space, if it holds that for each $L_1 \subset X$ metacompact, there is $L_2 \subset X$ closed metacompact, with $L_1 \subset L_2$ and there is $K \subset X$ connected set, with $K \subset X \setminus L_1$ and $K \cup L_2 = X$, and is denoted by semi-strong MJ-space.

Example 3.2: Let $(\mathbb{R}^2, \tau_u)$ be usual topological space. Assume that $D_n = \left(\bigcup_{i=1}^{\infty} D_{n,i}\right) \cup ([n, n+1] \times \{0\})$, where $i \geq 1$ and $n \geq 0$, Indeed, $D_{n,i}$ is the closed segment joining $(n, 0)$ to $(n+1, 1/i)$, and $[n, n+1] \times \{0\}$ is the horizontal closed segment joining $(n, 0)$ to $(n+1, 0)$. Let $X = \bigcup_{n=0}^{\infty} D_n$. Now, let $A_n = \{(x_1, x_2) \in X : x_1 \leq n\}$, we note that A_n is closed, since $A_n = \bigcup_{i=1}^{n-1} D_{n,i} \cup \{(n, 0)\}$, this means that A_n is finite union compact subsets, and thus A_n is compact, and thus A_n is metacompact. And assume that $B_n = \overline{(X \setminus A_n)}$ which is connected, in addition that $A_n \cup B_n = X$, now for every $K \subset X$ metacompact we can choose n such that $K \subset A_{n-1}$, and thus $K \subset A_n$ and $B_n \subset X \setminus K$. Hence X is a semi-strong MJ-space.

Example 3.3: Let $(\mathbb{N}, \tau)$ be a topological space, where $\mathbb{N}$ is the natural numbers set, and τ is the discrete topology, in this space if D is connected set, then D must to be singleton set. Now, to show that $(\mathbb{N}, \tau)$ does not represent semi-strong MJ-space, assume that $L_1 \subset \mathbb{N}$ is metacompact, and let $L_2 \subset \mathbb{N}$ is a closed metacompact such that $L_1 \subset L_2$ and $D \subset \mathbb{N}$ connected set, with $D \subset \mathbb{N} \setminus L_1$ and $D \cup L_2 = \mathbb{N}$, then D must be infinite, and this contradicts with the fact that D is connected. Thus $(\mathbb{N}, \tau)$ is not semi-strong MJ-space.

Definition 3.4: The space (X,τ) is defined as semi-weak metacompact J-space, if it holds that for each A_1 and A_2 are closed sets in (X,τ), with $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are metacompact, thus A_1 or A_2 is metacompact, and is denoted by semi-weak MJ-space.

Example 3.5: Let (X, τ) be a topological space, where X is infinite set, and τ is the co-finite topology, in this space every closed subset is compact, and thus it is metacompact D is connected set, then D must to be singleton set. Now, let A_1 and A_2 are closed sets in (X,τ), with $A_1 \cap A_2 = \emptyset$, since A_1 closed, so $\partial A_1 \subset A_1$, then ∂A_1 is metacompact, and in the same way, ∂A_2 is also metacompact, since both A_1 and A_2 are closed subset of X, hence A_1 and A_2 are metacompact. Therefore (X, τ) is a semi-weak MJ-space.

Definition 3.6: The space (X,τ) is defined as weak metacompact J-space, if it holds that for each $\{A_1, A_2, A_3\}$ constitutes a cover for (X,τ) by closed sets, with A_3 metacompact and $A_1 \cap A_2 = \emptyset$, then A_1 or A_2 is metacompact, and is denoted by weak MJ-space.

Example 3.7: Let (X, τ) be a space, where $\mathrm{X} = [-1,1]$, and $\tau = \{U \subset X | 0 \notin U \text{ } or \text{ } (-1,1) \subset U\}$, in this space the closed subsets are any set containing $\{0\}$, in addition $\{1\}, \{-1\}$ and $\{-1,1\}$. Now, let $\{A_1, A_2, A_3\}$ is a closed cover of (X,τ), such that A_3 metacompact and $A_1 \cap A_2 = \emptyset$, so A_1 or A_2 is finite, hence A_1 or A_2 is metacompact. Therefore (X, τ) is a weak MJ-space.

Theorem 3.8: Let (X, τ) be a topological space:

1. The (X, τ) is a strong MJ-space.
2. The (X, τ) is a semi-strong MJ-space.
3. The (X, τ) is an MJ-space.
4. The (X, τ) is a semi-weak MJ-space.
5. The (X, τ) is a weak MJ-space.

Hence (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5)

Proof: (1) $\Longrightarrow$ (2): Let L_1be a metacompact subset of (X, τ), but (X, τ) is a strong MJ-space, so there is metacompact closed set $L_2 \subset X$ with $L_1 \subset L_2$ with $X \setminus L_2$ is connected. Now let $K = X \setminus L_2$ which is connected, in addition that $K \subset X \setminus L_1$ because $L_1 \subset L_2$, and $K \cup L_2 = X$. Therefore (X, τ) is a semi-strong MJ-space.

(2) $\Longrightarrow$ (3): Let (X, τ) be a semi-strong MJ-space, and assume that $\{A_1, A_2\}$ constitutes a closed cover for (X, τ) such that $A_1 \cap A_2$ metacompact, thus there is a closed metacompact $A_3 \subset X$ with $A_1 \cap A_2 \subset A_3$, in addition that there is $A_4 \subset X$ connected set, with $A_4 \cup A_3 = X$ and $A_4 \subset X \setminus A_1 \cap A_2$ because (X, τ) is a semi-strong MJ-space. Indeed, $(A_1 \cap A_4) \cap (A_2 \cap A_4) = (A_1 \cap A_2) \cap A_4 = \emptyset$, since $A_4 \subset X \setminus A_1 \cap A_2$, in addition that $(A_1 \cap A_4) \cup (A_2 \cap A_4) = (A_1 \cup A_2) \cap A_4 = X \cap A_4 = A_4$, therefore $\{A_1 \cap A_4, A_2 \cap A_4\}$ is a cover of A_4 by disjoint closed sets, since A_4 is connected. So, either $A_4 \subset A_1 \cap A_4$ or $A_4 \subset A_2 \cap A_4$, and thus $A_4 \subset A_1$ or $A_4 \subset A_2$. If $A_4 \subset A_1$, then $A_4 \cap A_2 = \emptyset$, it follows that $A_2 \subset X \setminus A_4 \subset A_3$ which is a closed metacompact, so A_2 is metacompact. Similarly, if $A_4 \subset A_2$, then A_1 is metacompact. Hence (X, τ) is an MJ-space.

(3) $\Longrightarrow$ (4): Assume that (X, τ) is an MJ-space, and assume that A_1, A_2 is two closed sets in X with $\partial A_1, \partial A_2$ are metacompact and $A_1 \cap A_2 = \emptyset$, thus A_1 or A_2 is metacompact by Theorem (2.13) part 3. Hence (X, τ) is a semi-weak MJ-space.

(4) $\Longrightarrow$ (5): Assume that (X, τ) is a semi-weak MJ-space, and assume that $\{A_1, A_2, A_3\}$ is a cover of (X, τ) by closed sets with A_3 metacompact and $A_1 \cap A_2 = \emptyset$. Indeed, $\partial A_1 = \partial {A_1}^c$ and ${A_1}^c = A_2 \cup (A_3 \setminus A_3 \cap A_1)$, then $\partial A_1 = \partial\big(A_2 \cup (A_3 \setminus A_3 \cap A_1)\big)$, so $\partial A_1 \subset A_3 \cap A_1 \subset A_3$. By similar argument, one can show that $\partial A_2 \subset A_3$, therefore ∂A_1 and ∂A_2 are metacompact, hence by (4) A_1 or A_2 is metacompact. Therefore (X, τ) is a weak MJ-space.

Remark 3.9: If (X, τ) is a semi-strong MJ-space, then (X, τ) does not necessarily have to be strong MJ-space. The following example illustrates this.

Example 3.10: Let $(\mathbb{R}^2, \tau_u)$ be usual topological space. Assume $D_n = \left(\bigcup_{i=1}^{\infty} D_{n,i}\right) \cup ([n, n+1] \times \{0\})$, where $i \geq 1$ and $n \geq 0$. $X = \bigcup_{n=0}^{\infty} D_n$, from Example (3.2), we have that X is a semi-strong MJ-space. Now, to show that X is not strong MJ-space, assume that L is a closed metacompact subset of X, this means that L must contain a point on the x-axis, and thus $X \setminus L$ is not connected, hence X is not strong MJ-space.

Proposition 3.11: If (X, τ) is an MJ-space, and assume $Y = X \cup \{a\}$, then Y is a semi-weak MJ-space.

Proof: Assume that A_1 and A_2 are two closed sets in Y such that $\partial A_1, \partial A_2$ are metacompact and $A_1 \cap A_2 = \emptyset$, thus $a \notin A_1$ or $a \notin A_2$. Suppose that $a \notin A_2$, and assume that $D = \overline{(X \setminus A_2)}$, so $\{A_2, D\}$ is a cover of (X, τ) by closed sets with the property that $D \cap A_2 = \partial A_2$ which is metacompact, thus A_2 or D is metacompact because (X, τ) is an MJ-space. But $A_1 \subset D \cup \{a\}$, thus A_1 or A_2 is metacompact. Hence Y is a semi-weak MJ-space.

Remark 3.12: A semi-weak MJ-space need not be MJ-space. The following example illustrates this.

Example 3.13: Suppose that $X = \mathbb{R} \times [0,1)$, since X is a metric space, and thus X is metacompact, so by Proposition (2.12) and Proposition (2.10) X is an MJ-space. Assume that $Y = X \cup \{(0,1)\}$, since X is an MJ-space, so by Proposition (3.11) Y is a semi-weak MJ-space. Now, we need to show that Y is not MJ-space, assume that $K_1 = \{(a, b) \in Y : a \leq 0\}$ and $K_2 = \{(a, b) \in Y : a \geq 0\}$, we note that K_1 and K_2 are closed subsets of Y, in addition that $K_1 \cup K_2 = Y$, and $K_1 \cap K_2 = \{0\} \times [0,1) \cup \{(0,1)\} = \{0\} \times [0,1] = [0,1]$ which is compact, and thus it is metacompact, therefore $\{K_1, K_2\}$ is a closed cover of Y with $K_1 \cap K_2$ is metacompact, but neither K_1 nor K_2 is metacompact. Because there exists open cover of K_1 (or K_2) such that every neighborhood of $(0,1)$ intersects infinitely many members of the open cover, so there exists no a point-finite refinement for the open cover (Note that, any open neighborhood of $(0,1)$ necessarily contains points belonging to infinitely many horizontal slices at $b \to 1$). Therefore Y is not MJ-space.

Proposition 3.14: Assume that (X, τ) is a topological space, and let $\{X_1, X_2\}$ be a cover of (X, τ) by closed sets such that $X_1 \cap X_2$ is non-metacompact. If both X_1 and X_2 are weak MJ-space, then (X, τ) is a weak MJ-space.

Proof: Assume that $\{A^*, A', A\}$ is a cover of (X, τ) by closed sets with $A^* \cap A' = \emptyset$ and A is metacompact. To prove A^* or A' is metacompact, let ${A^*}_i = A^* \cap X_i$, ${A'}_i = A' \cap X_i$ and $A_i = A \cap X_i$, for $(i = 1,2)$. Then $\{{A^*}_i, {A'}_i, A_i\}$ is a closed cover of X_i such that ${A^*}_i \cap {A'}_i = \emptyset$ and A is metacompact. But X_1 is a weak MJ-space, consequently ${A'}_1$ or ${A^*}_1$ is metacompact. Let ${A'}_1$ be metacompact, we shall show that ${A'}_2$ is metacompact, if ${A'}_2$ is non-metacompact, thus ${A^*}_2$ must be metacompact, but X_2 is a weak MJ-space, consequently $G = {A^*}_2 \cup {A'}_1 \cup A$ is metacompact, since $(X_1 \cap X_2) \subset G$ closed set, thus $X_1 \cap X_2$ is metacompact which is a contradiction. Therefore $A' = {A'}_1 \cup {A'}_2$ is metacompact. Similarly, we can prove that A^* is metacompact whenever ${A^*}_1$ is metacompact.

Theorem 3.15: Assume that (X,τ) is a Hausdorff locally compact space, then the following statements are equivalent.

a) (X,τ) is an MJ-space.

b) (X,τ) is a semi-weak MJ-space.

c) (X,τ) is a weak MJ-space.

Proof: (a) ⟹ (b): Clear from Theorem (3.8).

(b) ⟹ (c): Clear from Theorem (3.8).

(c) ⟹ (a): Supoose that (X,τ) is a Hausdorff locally compact weak MJ-space, to show that (X,τ) is an MJ-space, assume that $\{G,D\}$ is a cover of (X,τ) by closed sets with $G\cap D$ metacompact. Since (X,τ) is a Hausdorff locally compact, thus $G\cap D\subset int(S)$, for some $S\subset X$ compact. Assume that $G^*=G\setminus int(S)$, and assume that $D^*=D\setminus int(S)$, so $\{G^*,D^*,S\}$ is a cover of (X,τ) by closed sets with S compact, and thus metacompact, and $G^*\cap D^*=\emptyset$, but (X,τ) is a weak MJ-space, then G^* or D^* is metacompact, therefore $G^*\cup S$ or $D^*\cup S$ is metacompact. Since $G\subset G^*\cup S$ and $D\subset D^*\cup S$ are closed sets, then G or D is metacompact by Theorem (2.8). Hence (X,τ) is an MJ-space.

4. The Relationship Between (Semi-Strong, Weak, Semi-Weak) MJ-spaces and (Semi-Strong, Weak, Semi- Weak) J- spaces

In this section we explain the relationship between the previous spaces.

Proposition 4.1: If a semi-strong MJ-space (X,τ) is countably compact, then (X,τ) is a semi-strong J-space.

Proof: Suppose that (X,τ) is a semi-strong MJ-space, to show that (X,τ) is a semi-strong J-space, let $L_1\subset X$ be a compact set, this means L_1 is metacompact in (X,τ), but (X,τ) is a semi-strong MJ-space, then there is a closed metacompact subset L_2 of (X,τ), with the property that L_1 contained in L_2 and there is a connected subset D of (X,τ) with $D\subset X\setminus L_1$ and $D\cup L_2=X$. Since $L_2\subset X$ is a closed set, and (X,τ) is countably compact, so L_2 is also countably compact, so L_2 is compact. Therefore (X,τ) is a semi-strong J-space.

Proposition 4.2: Assume that (X,τ) is a countably compact space, then (X,τ) is a semi-weak MJ-space if and only if (X,τ) is a semi-weak J-space.

Proof: Let (X,τ) be a semi-weak MJ-space, we need to prove that (X,τ) is a semi-weak J-space, assume that A_1 and A_2 is a closed in (X,τ) with $A_1\cap A_2=\emptyset$ and $\partial A_1,\partial A_2$ are compact, that is $\partial A_1,\partial A_2$ are metacompact, but (X,τ) is a semi-weak MJ-space, then A_1 or A_2 is metacompact, since A_1 and A_2 are closed subsets of (X,τ) which is countably compact, so A_1 and A_2 are countably compact, and so A_1 or A_2 is compact, then (X,τ) is a semi-weak J-space.

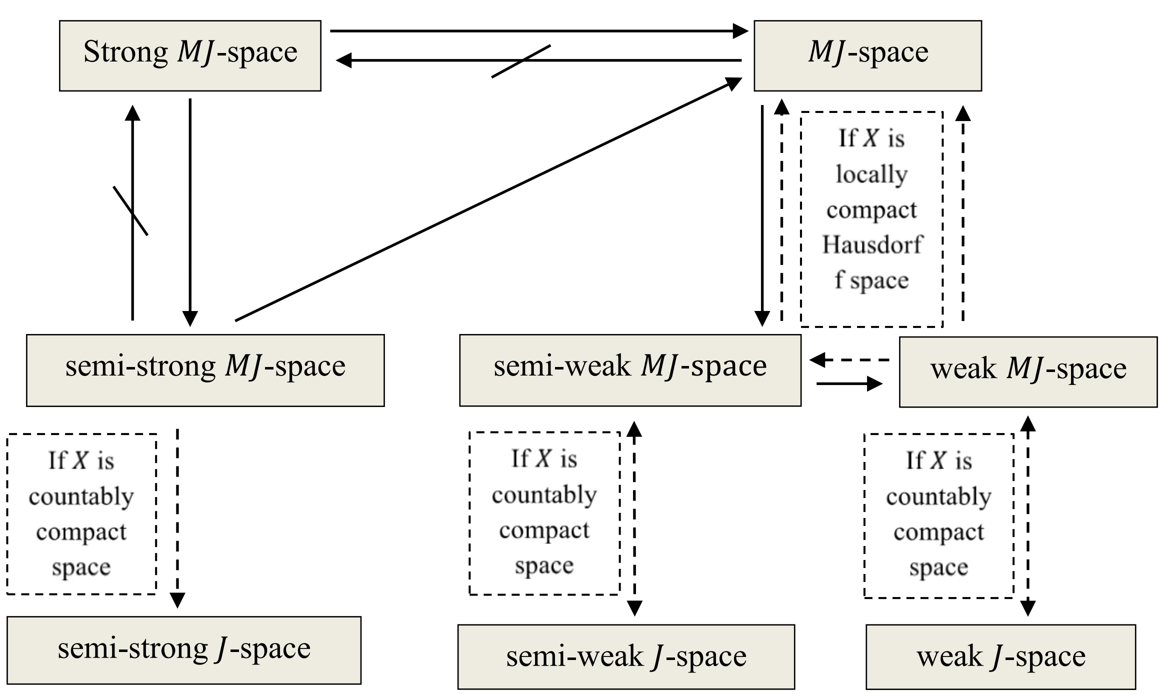
Now, assume that a semi-weak J-space (X,τ), to prove that (X,τ) is a semi-weak MJ-space, assume that A_1 and A_2 are a closed in (X,τ) such that $A_1 \cap A_2 = \emptyset$ and $\partial A_1, \partial A_2$ are metacompact, since ∂A_1 and ∂A_2 are closed subset of (X,τ), so ∂A_1 and ∂A_2 are countably compact, and thus ∂A_1 and ∂A_2 are compact sets, since we have (X,τ) is a semi-weak J-space, thus A_1 or A_2 is compact, and so A_1 or A_2 is metacompact, therefore (X,τ) is a semi-weak MJ-space.

Proposition 4.3: Assume that (X,τ) is a countably compact space, then (X,τ) is a weak MJ-space if and only if (X,τ) is a weak J-space.

Proof: Let (X,τ) be a weak MJ-space, we need to prove that (X,τ) is a weak J-space, assume that $\{A_1, A_2, A_3\}$ is a cover of (X,τ) by closed sets with A_3 is compact, in addition that $A_1 \cap A_2 = \emptyset$, that is A_3 is metacompact, but (X,τ) is a weak MJ-space, thus A_1 or A_2 is metacompact, since A_1 and A_2 are closed sets in (X,τ) which is countably compact, thus A_1 and A_2 are countably compact, and so A_1 or A_2 is compact, then (X,τ) is a weak J-space.

Now, assume that a weak J-space (X,τ), to show that (X,τ) is a weak MJ-space, let $\{A_1, A_2, A_3\}$ be a cover of (X,τ) by closed sets with A_3 metacompact and $A_1 \cap A_2 = \emptyset$, since $A_3 \subset X$ is a closed set, thus A_3 is countably compact, and thus A_3 is compact, but (X,τ) is a weak J-space, thus A_1 or A_2 is compact, hence A_1 or A_2 is metacompact, therefore (X,τ) is a weak MJ-space.

Now we will explain the relationship between the above spaces using the following diagram:



(Fig. 1) The relationship between the spaces

5. Conclusion

This work introduces the concept of semi-strong MJ-spaces, semi-weak MJ-spaces and weak MJ-spaces, as a new class of spaces closely related to MJ-spaces and strong MJ-spaces. In the third section of this paper, the relationship between these spaces was examined in Theorem (3.8), and Theorem (3.15) proved that MJ-spaces, semi-weak MJ-spaces and weak MJ-space are equivalent if a space satisfies the Hausdorff locally compact condition.

The fourth section explored the relationship between (semi-strong MJ-spaces, semi-weak MJ-spaces and weak MJ-spaces) and (semi-strong J-spaces, semi-weak J-spaces and weak J-spaces) respectively, where the Propositions (4.1, 4.2 and 4.3) demonstrated the crucial role of the countably compact condition in determining the relationship between these spaces.

For future research, this study can be expanded to explore further generalizations of MJ-spaces using other topological concepts.

References:

[2] C. Jordan, *Cours d'analyse de l'École polytechnique: t. Équations différentielles ordinaires. Équations linéaires. Équations aux dérivées partielles. Calcul des variations*. Gauthier-Villars et fils, 1887.

[3] R. A. H. a. H. S. Saeed, "Some New Generalized Types of " *F1000Research,* 2026.

[4] Y.-Z. J. C. M. J. Gao, "LJ-spaces," vol. 57, no. 4, pp. 1223-1237, 2007.

[5] S. S. Mthethwa and A. J. Q. M. Taherifar, "A note on J-spaces and relatively connected subsets," vol. 46, no. 11, pp. 2453-2464, 2023.

[6] S. S. Mthethwa and A. J. F. Taherifar, "J-spaces and C-normal spaces," vol. 38, no. 3, pp. 1009-1021, 2024.

[7] S. S. Mthethwa, S. J. C. Dubazana, and G. A. S. w. Applications, "On relatively connected sublocales and J-frames," vol. 22, no. 1, pp. 181-196, 2025.

[8] S. S. Dubazana and S. S. J. Q. M. Mthethwa, "On variants and generalizations of J-frames," vol. 48, no. 1, pp. 17-35, 2025.

[9] K. Morita and J.-I. Nagata, *Topics in general topology*. Elsevier, 1989.

[10] S. Soltanpour, "On Product of Metacompact Spaces," 2016.

[11] J. Oudetallah, M. M. Rousan, and I. M. J. J. A. M. I. Batiha, "On D-metacompactness in topological spaces," vol. 39, pp. 919-926, 2021.

[12] K. D. Joshi, *Introduction to general topology*. New Age International, 1983.

[13] J. L. Kelley, *General topology*. Courier Dover Publications, 2017.

[14] J. Derezinski, "General Topology," 2006.

[15] S. Willard, *General topology*. Courier Corporation, 2012.

[16] R. Engelking, "General Topology Heldermann," ed: Berlin, 1989.

[17] P. Howard, K. Keremedis, H. Rubin, and J. E. J. M. L. Q. Rubin, "Disjoint unions of topological spaces and choice," vol. 44, pp. 493-508, 1998.