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Some Fixed Point and Common Fixed-Point Results in Partial b-Metric Space

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Abstract:

In this research was our fundamental objective is to proving certain fixed point and common fixed-point results in partial b-metric space. The results that we obtained it was generalization and improvement some recent results of fixed-point theorems in partial b-metric space.

Mathematics Subject Classification: 47H10, 54H25

Key words: partial b-metric space (P. B. M. S), fixed point (F.P), common fixed point**.**

1. Introduction

The fixed-point theory is a very serious portion of a nonlinear-functional analysis. In 1922, Banach [3] introduced one of basic results about the field of (F.P) theory called Banach contraction principle (B. C. P) in complete metric spaces. There is a list of generalizations of metric space. One of these generalizations dates back to, S. Czerwik [6] in 1993 was introduced concept of b-metric space (B. M. S) that by multiplying the trigonometric inequality by the constant $b \geq 1$, he was proved a famous (B. C. P) in (B. M. S). As well many results have appeared in this filed, see ([14], [23], [7], [15]). In 1994, S. G. Matthews [16] present an idea of partial metric space (P. M. S) in it explained that the self-distance of any object not necessarily equal to the number zero. As well proved (B. C. P) in (P.M.S). A lot of (F.P) results were studied in this filed by other researchers see ([13], [18], [2], [4], [11]). In 2014, S. Shukla [19] present an idea of partial b metric space (P.B.M.S) and studied (B. C. P), Kannan condition in (P. B. M. S). As well many results have appeared in this filed, see ([17], [1], [20], [24], [8], [12], [22], [5], [21], [9], [10]). In this paper we prove new results of (F.P) and common (F.P) in (P. B. M. S). The results that we obtained was improvement and generalization some recent results of (F.P) theorems in (P. B. M. S).

2. Preliminaries:

Definition 2.1 [16] Assume Ψ a set that is non-empty, then a partial metric d_p on a set Ψ be a function $d_{\mathcal{P}}: \Psi^2 \to \mathbb{R}^+$ besides satisfying the following axioms;

(d_{p1}) $\tau = \kappa \leftrightarrow d_\rho(\tau,\kappa) = d_\rho(\tau,\tau) = d_\rho(\kappa,\kappa)$

(d_{p2}) $d_\rho(\tau,\tau) \le d_\rho(\tau,\kappa)$

(d_{p3}) $d_\rho(\tau,\kappa) = d_\rho(\kappa,\tau)$

(d_{p4}) $d_\rho(\tau,\kappa) \le d_\rho(\tau,h) + d_\rho(h,\kappa) - d_\rho(h,h)$

for all $\tau,\kappa,h \in \Psi$. Then (Ψ, d_ρ) called (P. M. S).

Definition 2.2 [6] Assume Ψ a set that is non-empty, then a b metric d_b on a set Ψ be a function $d_b: \Psi^2 \to \mathbb{R}^+$ besides satisfying the following axioms;

(d_{b1}) $d_b(\tau,\kappa) = 0 \leftrightarrow \tau = \kappa$

(d_{b2}) $d_b(\tau,\kappa) = d_b(\kappa,\tau)$

(d_{b3}) There exists a real number $b \ge 1$ such that $d_b(\tau,\kappa) \le b[d_b(\tau,h) + d_b(h,\kappa)]$

for all $\tau,\kappa,h \in \Psi$. Then (Ψ, d_b) called (B. M. S).

Definition 2.3 [19] Assume Ψ a set that is non-empty, then a partial b metric p on a set Ψ be a function $p: \Psi^2 \to \mathbb{R}^+$ besides satisfying the following axioms;

(d_{pb1}) $\tau = \kappa \leftrightarrow d_{pb}(\tau,\kappa) = d_{pb}(\tau,\tau) = d_{pb}(\kappa,\kappa)$

(d_{pb2}) $d_{pb}(\tau,\tau) \le d_{pb}(\tau,\kappa)$

(d_{pb3}) $d_{pb}(\tau,\kappa) = d_{pb}(\kappa,\tau)$

(d_{pb4}) $d_{pb}(\tau,\kappa) \le b[d_{pb}(\tau,h) + d_{pb}(h,\kappa)] - d_{pb}(h,h)$

for all $\tau,\kappa,h \in \Psi$. Then (Ψ, d_{pb}) called (P. B. M. S).

In (P. B. M. S), clearly $d_{pb}(\tau,\kappa) = 0$, by (d_{pb1}), (d_{pb2}) it follows $\tau = \kappa$. Whereas the converse not hold in general. Every b-Metric defined by Partial b-Metric a as following

$d_b(\tau,\kappa) = 2d_{pb}(\tau,\kappa) - d_{pb}(\tau,\tau) - d_{pb}(\kappa,\kappa)\ \forall \tau,\kappa \in \Psi$

Example 2.1 [12] Let $\Psi = \mathbb{R}^+$ and $d_{pb}: \Psi^2 \to \mathbb{R}^+$ be a function defined by $d_{pb}(\tau,\kappa) = [\max\{\tau,\kappa\}]^4 + |\tau - \kappa|^4\ \forall \tau,\kappa \in \Psi$**.** Then (Ψ, d_{pb}) is (P. B. M. S) with $b = 2^4 > 1$. but it is neither a (B. M. S) nor (P. M. S). Indeed, for any $\tau > 0$ we have $d_{pb}(\tau,\tau) = \tau^4 \neq 0$; therefore, d_{pb} is not a b-metric on Ψ. Also for $\tau = 6, \kappa = 2$, $h = 3$ we have $d_{pb}(\tau,\kappa) = 6^4 + 4^4$ and $d_{pb}(\tau,h) +$

$d_{pb}(h,\kappa) - d_{pb}(h,h) = 6^4 + 3^4 + 3^4 + 1^4 - 3^4 = 6^4 + 3^4 + 1^4$ so $d_{pb}(\tau,\kappa) > d_{pb}(\tau,h) + d_{pb}(h,\kappa) - d_{pb}(h,h)$, $\forall b = 2^4$. Therefore, d_{pb} is not a partial metric on Ψ.

Definition 2.4 [19] Let (Ψ, d_{pb}) be (P. B. M. S)

1- Assume that $\{\varpi_n\}$ be a sequence in (Ψ, d_{pb}) then $\{\varpi_n\}$ called convergent to the point $\varpi \in \Psi \Leftrightarrow \lim\limits_{n\to\infty} d_{pb}(\varpi_n, \varpi) = d_{pb}(\varpi, \varpi)$
2- Assume that $\{\varpi_n\}$ be a sequence in (Ψ, d_{pb}) then $\{\varpi_n\}$ called Cauchy $\Leftrightarrow \lim\limits_{m,n\to\infty} d_{pb}(\varpi_m, \varpi_n)$ exists (finite).
3- If for all Cauchy sequence $\{\varpi_n\}$ in Ψ $\exists \varpi \in \Psi$ such that $\lim\limits_{m,n\to\infty} d_{pb}(\varpi_n, \varpi_m) = \lim\limits_{n\to\infty} d_{pb}(\varpi_n, \varpi) = d_{pb}(\varpi, \varpi)$ then (Ψ, d_{pb}) called complete and denoted by (C. P. B. M. S) as short

3. Main results

Theorem 3.1 Assume that (Ψ, d_{pb}) a (C. P. B. M. S) with $b \geq 1$. Assume $T: \Psi \to \Psi$ be self-map satisfying the following condition,

$$\begin{aligned} d_{pb}(T\tau, T\kappa) \leq \Upsilon_1 d_{pb}(\tau,\kappa) &+ \Upsilon_2\left(\frac{[d_{pb}(\tau,T\tau) + d_{pb}(\kappa,T\kappa)][1 + d_{pb}(\tau,T\tau)]}{2[1 + d_{pb}(\tau,\kappa)]}\right) \\ &+ \Upsilon_3\left(\frac{[d_{pb}(\kappa,T\tau) + d_{pb}(\tau,T\kappa)][1 + d_{pb}(\tau,T\tau)]}{2[1 + d_{pb}(\tau,\kappa)]}\right) \\ &+ \Upsilon_4\left(\frac{[d_{pb}(\tau,\tau) + d_{pb}(\kappa,\kappa)][1 + d_{pb}(\tau,\kappa)]}{2[1 + d_{pb}(\tau,T\tau)]}\right) \end{aligned} \tag{3.1}$$

$\forall \tau, \kappa \in \Psi$, $\Upsilon_1, \Upsilon_2, \Upsilon_{3,} \Upsilon_4 \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b + \Upsilon_4 < 1$. Then T has unique (F.P).

Proof: Let $\varpi_0 \in \Psi$ we define sequence $\{\varpi_n\}$ in Ψ by denoting $\varpi_{n+1} = T\varpi_n$. If $\varpi_n = \varpi_{n+1}$ for some $n \in N$ then $\varpi_n = T\varpi_n$ is (F.P) of T. Suppose $\varpi_n \neq \varpi_{n+1}$ for all $n \in N$ and using (3.1)

$$\begin{aligned} d_{pb}(\varpi_n, \varpi_{n+1}) = {} & d_{pb}(T\varpi_{n-1}, T\varpi_n) \\ & \leq \Upsilon_1 d_{pb}(\varpi_{n-1}, \varpi_n) \\ & + \Upsilon_2\left(\frac{[d_{pb}(\varpi_{n-1}, T\varpi_{n-1}) + d_{pb}(\varpi_n, T\varpi_n)][1 + d_{pb}(\varpi_{n-1}, T\varpi_{n-1})]}{2[1 + d_{pb}(\varpi_{n-1}, \varpi_n)]}\right) \\ & + \Upsilon_3\left(\frac{[d_{pb}(\varpi_n, T\varpi_{n-1}) + d_{pb}(\varpi_{n-1}, T\varpi_n)][1 + d_{pb}(\varpi_{n-1}, T\varpi_{n-1})]}{2[1 + d_{pb}(\varpi_{n-1}, \varpi_n)]}\right) \\ & + \Upsilon_4\left(\frac{[d_{pb}(\varpi_{n-1}, \varpi_{n-1}) + d_{pb}(\varpi_n, \varpi_n)][1 + d_{pb}(\varpi_{n-1}, \varpi_n)]}{2[1 + d_{pb}(\varpi_{n-1}, T\varpi_{n-1})]}\right) \end{aligned}$$

$$d_{pb}(\varpi_n,\varpi_{n+1})$$
$$\leq \Upsilon_1 d_{pb}(\varpi_{n-1},\varpi_n)$$
$$+\Upsilon_2\left(\frac{\left[d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right]\left[1+d_{pb}(\varpi_{n-1},\varpi_n)\right]}{2\left[1+d_{pb}(\varpi_{n-1},\varpi_n)\right]}\right)$$
$$+\Upsilon_3\left(\frac{\left[d_{pb}(\varpi_n,\varpi_n)+d_{pb}(\varpi,\varpi_{n+1})\right]\left[1+d_{pb}(\varpi_{n-1},\varpi_n)\right]}{2\left[1+d_{pb}(\varpi_{n-1},\varpi_n)\right]}\right)$$
$$+\Upsilon_4\left(\frac{\left[d_{pb}(\varpi_{n-1},\varpi_{n-1)}+d_{pb}(\varpi_n,\varpi_n)\right]\left[1+d_{pb}(\varpi_{n-1},\varpi_n)\right]}{2\left[1+d_{pb}(\varpi,\varpi_n)\right]}\right)$$

$$d_{pb}(\varpi_n,\varpi_{n+1})$$
$$\leq \Upsilon_1 d_{pb}(\varpi_{n-1},\varpi_n)+\frac{\Upsilon_2}{2}\left(d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right)$$
$$+\frac{\Upsilon_3}{2}\left(d_{pb}(\varpi_n,\varpi_n)+d_{pb}(\varpi_{n-1},\varpi_{n+1})\right)$$
$$+\frac{\Upsilon_4}{2}\left(d_{pb}(\varpi_{n-1},\varpi_{n-1})+d_{pb}(\varpi_n,\varpi_n)\right)$$

By using (d_{pb2}) and (d_{pb4})

$$d_{pb}(\varpi_n,\varpi_{n+1})$$
$$\leq \Upsilon_1 d_{pb}(\varpi_{n-1},\varpi_n)+\frac{\Upsilon_2}{2}\left(d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right)$$
$$+\frac{\Upsilon_3}{2}\left(d_{pb}(\varpi_n,\varpi_n)+b\left[d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right]-d_{pb}(\varpi_n,\varpi_n)\right)$$
$$+\frac{\Upsilon_4}{2}\left(d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_{n-1},\varpi_n)\right)$$

$$d_{pb}(\varpi_n,\varpi_{n+1})$$
$$\leq \Upsilon_1 d_{pb}(\varpi_{n-1},\varpi_n)+\frac{\Upsilon_2}{2}\left(d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right)$$
$$+\frac{\Upsilon_3}{2}\left(b\left[d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_n,\varpi_{n+1})\right]\right)$$
$$+\frac{\Upsilon_4}{2}\left(d_{pb}(\varpi_{n-1},\varpi_n)+d_{pb}(\varpi_{n-1},\varpi_n)\right)$$

$$\left(1-\frac{\Upsilon_2}{2}-\frac{\Upsilon_3 b}{2}\right)d_{pb}(\varpi,\varpi_{n+1})\leq\left(\Upsilon_1+\frac{\Upsilon_2}{2}+\frac{\Upsilon_3 b}{2}+\Upsilon_4\right)d_{pb}(\varpi_{n-1},\varpi_n)$$

$$d_{pb}(\varpi_n,\varpi_{n+1})\leq\frac{\left(\Upsilon_1+\frac{\Upsilon_2}{2}+\frac{\Upsilon_3 b}{2}+\Upsilon_4\right)}{\left(1-\frac{\Upsilon_2}{2}-\frac{\Upsilon_3 b}{2}\right)}d_{pb}(\varpi_{n-1},\varpi_n),$$

Putting $\beta=\frac{\left(\Upsilon_1+\frac{\Upsilon_2}{2}+\frac{\Upsilon_3 b}{2}+\Upsilon_4\right)}{\left(1-\frac{\Upsilon_2}{2}-\frac{\Upsilon_3 b}{2}\right)}$, $0<\beta<1$

$$d_{pb}(\varpi_n, \varpi_{n+1}) \le \beta d_{pb}(\varpi_{n-1}, \varpi_n) \le \beta^2 d_{pb}(\varpi_{n-2}, \varpi_{n-1}) \le \beta^3 d_{pb}(\varpi_{n-3}, \varpi_{n-2}) \le \cdots \le \beta^n d_{pb}(\varpi_0, \varpi_1)$$

$$\lim_{n\to\infty} d_{pb}(\varpi_n, \varpi_{n+1}) = 0$$

$$\begin{aligned} d_{pb}(\varpi_n, \varpi_m) &\le b\big[d_{pb}(\varpi_n, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, \varpi_m)\big] - d_{pb}(\varpi_{n+1}, \varpi_{n+1}) \\ &\le b\big[d_{pb}(\varpi_n, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, \varpi_m)\big] \\ &\le bd_{pb}(\varpi_n, \varpi_{n+1}) \\ &\quad + b\left(b\big[d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + d_{pb}(\varpi_{n+2}, \varpi_m)\big] - d_{pb}(\varpi_{n+2}, \varpi_{n+2})\right) \\ &\le bd_{pb}(\varpi, \varpi_{n+1}) + b^2 d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + b^2 d_{pb}(\varpi_{n+2}, \varpi_m) \\ &\vdots \\ &\le bd_{pb}(\varpi_n, \varpi_{n+1}) + b^2 d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + b^3 d_{pb}(\varpi_{n+2}, \varpi_{n+3}) + \cdots \\ &\quad + b^{m-n} d_{pb}(\varpi_{m-1}, \varpi_m) \\ &\le b\beta^n d_{pb}(\varpi_0, \varpi_1) + b^2\beta^{n+1} d_{pb}(\varpi_0, \varpi_1) + b^3\beta^{n+2} d_{pb}(\varpi_0, \varpi_1) + \cdots \\ &\quad + b^{m-n}\beta^{m-1} d_{pb}(\varpi_0, \varpi_1) \\ &\le b\beta^n (1 + b\beta + (b\beta)^2 + \cdots + (b\beta)^{m-n-1}) d_{pb}(\varpi_0, \varpi_1) \end{aligned}$$

$\lim_{n,m\to\infty} d_{pb}(\varpi_n, \varpi_m) = 0$. Thus (from Definition 2.4 (2)) $\{\varpi_n\}$ be Cauchy sequence in Ψ. Since (Ψ, d_{pb}) is complete (P. B. M. S) then (from Definition 2.4 (3)) $\exists \varpi \in \Psi$ s.t $\lim_{n,m\to\infty} d_{pb}(\varpi_n, \varpi_m) = \lim_{n\to\infty} d_{pb}(\varpi_n, \varpi) = d_{pb}(\varpi, \varpi) = 0$

We need to show that ϖ is (F. P) of T

$$\begin{aligned} d_{pb}(\varpi, T\varpi) &\le b\big[d_{pb}(\varpi, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, T\varpi)\big] - d_{pb}(\varpi_{n+1}, \varpi_{n+1}) \\ &\le b\big[d_{pb}(\varpi, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, T\varpi)\big] \\ &\le bd_{pb}(\varpi, \varpi_{n+1}) + bd_{pb}(T\varpi_n, T\varpi) \end{aligned}$$

$$\leq bd_{pb}(\varpi,\varpi_{n+1})$$
$$+b\left(\Upsilon_1 d_{pb}(\varpi_n,\varpi)+\Upsilon_2\left(\frac{[d_{pb}(\varpi_n,T\varpi_n)+d_{pb}(\varpi,T\varpi)][1+d_{pb}(\varpi_n,T\varpi_n)]}{2[1+d_{pb}(\varpi_n,\varpi)]}\right)\right.$$
$$+\Upsilon_3\left(\frac{[d_{pb}(\varpi,T\varpi_n)+d_{pb}(\varpi_n,T\varpi)][1+d_{pb}(\varpi_n,T\varpi_n)]}{2[1+d_{pb}(\varpi_n,\varpi)]}\right)$$
$$\left.+\Upsilon_4\left(\frac{[d_{pb}(\varpi_n,\varpi_n)+d_{pb}(\varpi,\varpi)][1+d_{pb}(\varpi_n,\varpi)]}{2[1+d_{pb}(\varpi,T\varpi_n)]}\right)\right)$$

$$\leq bd_{pb}(\varpi,\varpi_{n+1})$$
$$+b\left(\Upsilon_1 d_{pb}(\varpi_n,\varpi)\right.$$
$$+\Upsilon_2\left(\frac{[d_{pb}(\varpi_n,\varpi_{n+1})+d_{pb}(\varpi,T\varpi)][1+d_{pb}(\varpi_n,\varpi_{n+1})]}{2[1+d_{pb}(\varpi_n,\varpi)]}\right)$$
$$+\Upsilon_3\left(\frac{[d_{pb}(\varpi,\varpi_{n+1})+d_{pb}(\varpi_n,T\varpi)][1+d_{pb}(\varpi_n,\varpi_{n+1})]}{2[1+d_{pb}(\varpi_n,\varpi)]}\right)$$
$$\left.+\Upsilon_4\left(\frac{[d_{pb}(\varpi_n,\varpi_n)+d_{pb}(\varpi,\varpi)][1+d_{pb}(\varpi_n,\varpi)]}{2[1+d_{pb}(\varpi_n,\varpi_{n+1})]}\right)\right)$$

By take limit when $n \to \infty$ we have $d_{pb}(\varpi,T\varpi)\leq 0$. Therefore $d_{pb}(\varpi,T\varpi)=0$ then $\varpi=T\varpi$ hence ϖ is (F.P) of T

Uniqueness: Suppose that $\omega,\varpi\in\Psi$ are two (F. P) of T s.t $\omega\neq\varpi$

$$d_{pb}(\omega,\varpi)=d_{pb}(T\omega,T\varpi)$$
$$\leq \Upsilon_1 d_{pb}(\omega,\varpi)+\Upsilon_2\left(\frac{[d_{pb}(\omega,T\omega)+d_{pb}(\varpi,T\varpi)][1+d_{pb}(\omega,T\omega)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_3\left(\frac{[d_{pb}(\varpi,T\omega)+d_{pb}(\omega,T\varpi)][1+{}_{pb}(\omega,T\omega)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_4\left(\frac{[d_{pb}(\omega,\omega)+d_{pb}(\varpi,\varpi)][1+d_{pb}(\omega,\varpi)]}{2[1+{}_{pb}(\omega,T\omega)]}\right)$$

$$\leq \Upsilon_1 d_{pb}(\omega,\varpi) + \Upsilon_2\left(\frac{[d_{pb}(\omega,\omega) + d_{pb}(\varpi,\varpi)][1 + d_{pb}(\omega,\omega)]}{2[1 + d_{pb}(\omega,\varpi)]}\right)$$
$$+ \Upsilon_3\left(\frac{[d_{pb}(\varpi,\omega) + d_{pb}(\omega,\varpi)][1 + d_{pb}(\omega,\omega)]}{2[1 + d_{pb}(\omega,\varpi)]}\right)$$
$$+ \Upsilon_4\left(\frac{[d_{pb}(\omega,\omega) + d_{pb}(\varpi,\varpi)][1 + d_{pb}(\omega,\varpi)]}{2[1 + d_{pb}(\omega,\omega)]}\right)$$

Since $d_{pb}(\omega,\omega) \leq d_{pb}(\omega,\varpi)$, $d_{pb}(\varpi,\varpi) \leq d_{pb}(\omega,\varpi)$ $\forall \omega,\varpi \in \Psi$

$$d_{pb}(\omega,\varpi) \leq \Upsilon_1 d_{pb}(\omega,\varpi) + \Upsilon_2\left(\frac{[d_{pb}(\omega,\varpi) + d_{pb}(\omega,\varpi)][1 + d_{pb}(\omega,\omega)]}{2[1 + d_{pb}(\omega,\varpi)]}\right)$$
$$+ \Upsilon_3\left(\frac{[d_{pb}(\varpi,\omega) + d_{pb}(\omega,\varpi)][1 + d_{pb}(\omega,\varpi)]}{2[1 + d_{pb}(\omega,\omega^{\wedge})]}\right)$$
$$+ \Upsilon_4\left(\frac{[d_{pb}(\omega,\varpi) + d_{pb}(\omega,\varpi)][1 + d_{pb}(\omega,\varpi)]}{2[1 + d_{pb}(\omega,\varpi)]}\right)$$

$$d_{pb}(\omega,\varpi) \leq \Upsilon_1 d_{pb}(\omega,\varpi) + \Upsilon_2 d_{pb}(\omega,\varpi) + \Upsilon_3 d_{pb}(\omega,\varpi) + \Upsilon_4 d_{pb}(\omega,\varpi)$$

$$(1 - \Upsilon_1 - \Upsilon_2 - \Upsilon_3 - \Upsilon_4) d_{pb}(\omega,\varpi) \leq 0$$

$d_{pb}(\omega,\varpi) \leq 0$. Therefore $d_{pb}(\omega,\varpi) = 0$. Thus $\omega = \varpi$ contradiction.

Corollary 3.1 Assume that (Ψ, d_{pb}) a (C. P. B. M. S) with $b \geq 1$. Assume $T: \Psi \to \Psi$ be self-map satisfying the following condition,

$$d_{pb}(T\tau,T\kappa) \leq \Upsilon_1 d_{pb}(\tau,\kappa) + \Upsilon_2\left(\frac{[d_{pb}(\tau,T\tau) + d_{pb}(\kappa,T\kappa)][1 + d_{pb}(\tau,T\tau)]}{2[1 + d_{pb}(\tau,\kappa)]}\right)$$
$$+ \Upsilon_3\left(\frac{[d_{pb}(\kappa,T\tau) + d_{pb}(\tau,T\kappa)][1 + d_{pb}(\tau,T\tau)]}{2[1 + d_{pb}(\tau,\kappa)]}\right) \tag{3.2}$$

$\forall \tau,\kappa \in \Psi$, $\Upsilon_1, \Upsilon_2, \Upsilon_3 \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b < 1$. Then T has unique (F. P).

Corollary 3.2 Assume that (Ψ, d_{pb}) a (C. P. B. M. S) with $b \geq 1$. Assume $T: \Psi \to \Psi$ be self-map satisfying the following condition,

$$d_{pb}(T\tau, T\kappa) \leq \Upsilon_1 d_{pb}(\tau, \kappa) + \Upsilon_2 \left(\frac{[d_{pb}(\tau, T\tau) + d_{pb}(\kappa, T\kappa)][1 + d_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) + \Upsilon_3 \left(\frac{[d_{pb}(\tau, \tau) + d_{pb}(\kappa, \kappa)][1 + d_{pb}(\tau, \kappa)]}{2[1 + d_{pb}(\tau, T\tau)]} \right) \quad (3.3)$$

$\forall \tau, \kappa \in \Psi$, $\Upsilon_1, \Upsilon_2, \Upsilon_3 \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 < 1$. Then T has unique (F. P).

Corollary 3.3 Assume that (Ψ, d_{pb}) a (C. P. B. M. S) with $b \geq 1$. Assume $T: \Psi \to \Psi$ be self-map satisfying the following condition,

$$d_{pb}(T\tau, T\kappa) \leq \Upsilon_1 d_{pb}(\tau, \kappa) + \Upsilon_2 \left(\frac{[d_{pb}(\kappa, T\tau) + d_{pb}(\tau, T\kappa)][1 + d_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) + \Upsilon_3 \left(\frac{[d_{pb}(\tau, \tau) + d_{pb}(\kappa, \kappa)][1 + d_{pb}(\tau, \kappa)]}{2[1 + d_{pb}(\tau, T\tau)]} \right) \quad (3.4)$$

$\forall \tau, \kappa \in \Psi$, $\Upsilon_1, \Upsilon_2, \Upsilon_{3,} \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b < 1$. Then T has unique (F. P).

Corollary 3.4 Assume that (Ψ, d_{pb}) a (C. P. B. M. S) with $b \geq 1$. Assume $T: \Psi \to \Psi$ be self-map satisfying the following condition,

$$d_{pb}(T\tau, T\kappa) \leq \nu d_{pb}(\tau, \kappa) \quad (3.5)$$

$\forall \tau, \kappa \in \Psi$, and $0 < \Upsilon < 1$. Then T has unique (F.P).

Theorem 3.2 Assume (Ψ, d_{pb}) a complete (P. B. M. S) with $b \geq 1$. Assume $T, F: \Psi \to \Psi$ self-map satisfying the following condition,

$$d_{pb}(T\tau, F\kappa) \leq \Upsilon_1 d_{pb}(\tau, \kappa) + \Upsilon_2 \left(\frac{[d_{pb}(\tau, T\tau) + d_{pb}(\kappa, F\kappa)][1 + {}_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, n)]} \right) + \Upsilon_3 \left(\frac{[d_{pb}(\kappa, T\tau) + d_{pb}(\tau, F\kappa)][1 + {}_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) + \Upsilon_4 \left(\frac{[d_{pb}(\tau, \tau) + d_{pb}(\kappa, \kappa)][1 + d_{pb}(\tau, \kappa)]}{2[1 + {}_{pb}(\tau, T\tau)]} \right) \quad (3.6)$$

$\forall \tau, \kappa \in \Psi, \Upsilon_1, \Upsilon_2, \Upsilon_{3,} \Upsilon_4 \geq 0, 0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b + \Upsilon_4 < 1$. Then T, F have unique common (F.P).

Proof: Let $\omega_0 \in \Psi$ we define sequence $\{\omega_n\}$ in Ψ by denoting $\omega_{2n+1} = T\omega_{2n}$ and $\omega_{2n+2} = F\omega_{2n+1}$.

$$
\begin{aligned}
d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) = \ & d_{pb}(T\varpi_{2n}, F\varpi_{2n+1}) \\
& \leq \Upsilon_1 d_{pb}(\varpi_{2n}, \varpi_{2n+1}) \\
& + \Upsilon_2 \left(\frac{[d_{pb}(\varpi_{2n}, T\varpi_{2n}) + d_{pb}(\varpi_{2n+1}, F\varpi_{2n+1})][1 + d_{pb}(\varpi_{2n}, T\varpi_{2n})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]} \right) \\
& + \Upsilon_3 \left(\frac{[d_{pb}(\varpi_{2n+1}, T\varpi_{2n}) + d_{pb}(\varpi_{2n}, F\varpi_{2n+1})][1 + d_{pb}(\varpi_{2n}, T\varpi_{2n})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]} \right) \\
& + \Upsilon_4 \left(\frac{[d_{pb}(\varpi_{2n}, \varpi_{2n}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+1})][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, T\varpi_{2n})]} \right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \Upsilon_1 d_{pb}(\varpi_{2n}, \varpi_{2n+1}) \\
& + \Upsilon_2 \left(\frac{[d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+2})][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]} \right) \\
& + \Upsilon_3 \left(\frac{[d_{pb}(\varpi_{2n+1}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n}, \varpi_{2n+2})][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]} \right) \\
& + \Upsilon_4 \left(\frac{[d_{pb}(\varpi, \varpi_{2n}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+1})][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]} \right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \Upsilon_1 d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + \frac{\Upsilon_2}{2}\left(d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \right) \\
& + \frac{\Upsilon_3}{2}\left(d_{pb}(\varpi_{2n+1}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n}, \varpi_{2n+2}) \right) \\
& + \frac{\Upsilon_4}{2}\left(d_{pb}(\varpi_{2n}, \varpi_{2n}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+1}) \right)
\end{aligned}
$$

By using (d_{pb2}) and (d_{pb4})

$$
\begin{aligned}
& d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \\
& \leq \Upsilon_1 d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + \frac{\Upsilon_2}{2}\left(d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \right) \\
& + \frac{\Upsilon_3}{2}\left(d_{pb}(\varpi_{2n+1}, \varpi_{2n+1}) + b[d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+2})] \right. \\
& \left. - d_{pb}(\varpi_{2n+1}, \varpi_{2n+1}) \right) + \frac{\Upsilon_4}{2}\left(d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi_{2n}, \varpi_{2n+1}) \right)
\end{aligned}
$$

$$
(1 - \frac{\Upsilon_2}{2} - \frac{\Upsilon_3 b}{2}) d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \leq (\Upsilon_1 + \frac{\Upsilon_2}{2} + \frac{\Upsilon_3 b}{2} + \Upsilon_4) d_{pb}(\varpi_{2n}, \varpi_{2n+1})
$$

$$
d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \leq \frac{(\Upsilon_1 + \frac{\Upsilon_2}{2} + \frac{\Upsilon_3 b}{2} + \Upsilon_4)}{(1 - \frac{\Upsilon_2}{2} - \frac{\Upsilon_3 b}{2})} \ d_{pb}(\varpi_{2n}, \varpi_{2n+1}),
$$

Putting $\beta = \frac{(\Upsilon_1+\frac{\Upsilon_2}{2}+\frac{\Upsilon_3 b}{2}+\Upsilon_4)}{(1-\frac{\Upsilon_2}{2}-\frac{\Upsilon_3 b}{2})}$ $, 0 \leq \beta < 1$

$$d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) \leq \beta\, d_{pb}(\varpi_{2n}, \varpi_{2n+1}) \leq \beta^2 d_{pb}(\varpi_{2n-1}, \varpi_{2n}) \leq \beta^3 d_{pb}(\varpi_{2n-2}, \varpi_{2n-1})$$
$$\leq \cdots \leq \beta^n d_{pb}(\varpi_0, \varpi_1)$$

$$\lim_{n\to\infty} d_{pb}(\varpi_{2n+1}, \varpi_{2n+2}) = 0$$

In general, $d_{pb}(\varpi_n, \varpi_{n+1}) \leq \beta^n d_{pb}(\varpi_0, \varpi_1)$

$$\begin{aligned} d_{pb}(\varpi_n, \varpi_m) &\leq b\left[d_{pb}(\varpi_n, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, \varpi_m)\right] - d_{pb}(\varpi_{n+1}, \varpi_{n+1}) \\ &\leq b\left[d_{pb}(\varpi_n, \varpi_{n+1}) + d_{pb}(\varpi_{n+1}, \varpi_m)\right] \\ &\leq b d_{pb}(\varpi_n, \varpi_{n+1}) \\ &+ b\left(b\left[d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + d_{pb}(\varpi_{n+2}, \varpi_m)\right] - d_{pb}(\varpi_{n+2}, \varpi_{n+2})\right) \\ &\leq b d_{pb}(\varpi_n, \varpi_{n+1}) + b^2 d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + b^2 d_{pb}(\varpi_{n+2}, \varpi_m) \\ &\vdots \\ &\leq b d_{pb}(\varpi_n, \varpi_{n+1}) + b^2 d_{pb}(\varpi_{n+1}, \varpi_{n+2}) + b^3 d_{pb}(\varpi_{n+2}, \varpi_{n+3}) + \cdots \\ &+ \qquad b^{m-n} d_{pb}(\varpi_{m-1}, \varpi_m) \\ &\leq b\beta^n d_{pb}(\varpi_0, \varpi_1) + b^2\beta^{n+1} d_{pb}(\varpi_0, \varpi_1) + b^3\beta^{n+2} d_{pb}(\varpi_0, \varpi_1) + \cdots \\ &+ b^{m-n}\beta^{m-1} d_{pb}(\varpi_0, \varpi_1) \\ &\leq b\beta^n(1 + b\beta + (b\beta)^2 + \cdots + (b\beta)^{m-n-1}) d_{pb}(\varpi_0, \varpi_1) \end{aligned}$$

$\lim_{n,m\to\infty} d_{pb}(\varpi_n, \varpi_m) = 0$. Thus (from Definition 2.4 (2)) $\{\varpi_n\}$ be Cauchy sequence in Ψ. Since Ψ is complete (P. B. B. S) then (from Definition 2.4 (3)) $\exists \varpi \in \Psi$ s.t

$$\lim_{n,m\to\infty} d_{pb}(\varpi_n, \varpi_m) = \lim_{n\to\infty} d_{pb}(\varpi_n, \varpi) = d_{pb}(\varpi, \varpi) = 0$$

We need to show that ϖ is common (F. P) of T, F

$$\begin{aligned} d_{pb}(\varpi, T\varpi) &\leq b\left[d_{pb}(\varpi, F\varpi_{2n+1}) + d_{pb}(F\varpi_{2n+1}, T\varpi)\right] - d_{pb}(F\varpi_{2n+1}, F\varpi_{2n+1}) \\ &\leq b\left[d_{pb}(\varpi, F\varpi_{2n+1}) + d_{pb}(F\varpi_{2n+1}, T\varpi)\right] \\ &\leq b d_{pb}(\varpi, F\varpi_{2n+1}) + b d_{pb}(F\varpi_{2n+1}, T\varpi) \\ &\leq b d_{pb}(\varpi, F\varpi_{2n+1}) + b d_{pb}(T\varpi, F\varpi_{2n+1}) \end{aligned}$$

$$
\begin{aligned}
&\leq bd_{pb}(\varpi, F\varpi) \\
&\quad + b\Bigg[\Upsilon_1 d_{pb}(\varpi, \varpi_{2n+1}) \\
&\quad + \Upsilon_2\left(\frac{[d_{pb}(\varpi, T\varpi) + d_{pb}(\varpi_{2n+1}, F\varpi_{2n+1})][1 + d_{pb}(\varpi, T\varpi)]}{2[1 + d_{pb}(\varpi, \varpi_{2n+1})]}\right)\Bigg] \\
&\quad + \Upsilon_3\left(\frac{[d_{pb}(\varpi, F\varpi_{2n+1}) + d_{pb}(\varpi_{2n+1}, T\varpi)][1 + d_{pb}(\varpi, T\varpi)]}{2[1 + d_{pb}(\varpi, \varpi_{2n+1})]}\right) \\
&\quad + \Upsilon_4\left(\frac{[d_{pb}(\varpi, \varpi) + d_{pb}(\varpi_{2n+1}, \varpi_{2n+1})][1 + d_{pb}(\varpi, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi, T\varpi)]}\right)
\end{aligned}
$$

By take limit when $n \to \infty$ then $d_{pb}(\varpi, T\varpi) \leq 0$. Therefore $d_{pb}(\varpi, T\varpi) = 0.$ Hence $\varpi = T\varpi$

$$
\begin{aligned}
d_{pb}(\varpi, F\varpi) &\leq b[d_{pb}(\varpi, T\varpi_{2n}) + d_{pb}(T\varpi_{2n}, F\varpi)] - d_{pb}(T\varpi_{2n}, T\varpi) \\
&\leq b[d_{pb}(\varpi, T\varpi_{2n}) + d_{pb}(T\varpi_{2n}, F\varpi)] \\
&\leq bd_{pb}(\varpi, T\varpi_{2n}) + bd_{pb}(T\varpi_{2n}, F\varpi) \\
&\leq bd_{pb}(\varpi, \varpi_{2n+1}) \\
&\quad + b\Bigg[\Upsilon_1 d_{pb}(\varpi_{2n}, \varpi) \\
&\quad + \Upsilon_2\left(\frac{[d_{pb}(\varpi_{2n}, \varpi_{2n+1}) + d_{pb}(\varpi, F\varpi)][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi)]}\right) \\
&\quad + \Upsilon_3\left(\frac{[d_{pb}(\varpi_{2n}, F\varpi) + d_{pb}(\varpi, \varpi_{2n+1})][1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}{2[1 + d_{pb}(\varpi_{2n}, \varpi)]}\right) \\
&\quad + \Upsilon_4\left(\frac{[d_{pb}(\varpi_{2n}, \varpi_{2n}) + d_{pb}(\varpi, \varpi)][1 + d_{pb}(\varpi_{2n}, \varpi)]}{2[1 + d_{pb}(\varpi_{2n}, \varpi_{2n+1})]}\right)\Bigg]
\end{aligned}
$$

By take limit when $n \to \infty$ we have $d_{pb}(\varpi, F\varpi) \leq 0$.

Therefore $d_{pb}(\varpi, F\varpi) = 0.$ hence $\varpi = F\varpi$. Then ϖ is common (F. P) of T and F

Uniqueness

Suppose that $\omega,\varpi \in \Psi$ are two (F. P) of T and F s.t $T\omega = F\varpi = \varpi, T\omega = F\varpi = \omega$ and $\omega \neq \varpi$

$$d_{pb}(\omega,\varpi)=d_{pb}(T\omega,F\varpi)$$
$$\leq \Upsilon_1 d_{pb}(\omega,\varpi)+\Upsilon_2\left(\frac{[d_{pb}(\omega,T\omega)+d_{pb}(\varpi,F\varpi)][1+d_{pb}(\omega,T\omega)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_3\left(\frac{[d_{pb}(\varpi,T\omega)+d_{pb}(\omega,F\varpi)][1+d_{pb}(\omega,T\omega)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_4\left(\frac{[d_{pb}(\omega,\omega)+d_{pb}(\varpi,\varpi)][1+d_{pb}(\omega,\varpi)]}{2[1+d_{pb}(\omega,T\omega)]}\right)$$

$$d_{pb}(\omega,\varpi)\leq \Upsilon_1 d_{pb}(\omega,\varpi)+\Upsilon_2\left(\frac{[d_{pb}(\omega,\varpi)+d_{pb}(\omega,\varpi)][1+d_{pb}(\omega,\varpi)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_3\left(\frac{[d_{pb}(\omega,\varpi)+d_{pb}(\omega,\varpi)][1+d_{pb}(\omega,\varpi)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$
$$+\Upsilon_4\left(\frac{[d_{pb}(\omega,\omega)+d_{pb}(\varpi,\varpi)][1+d_{pb}(\omega,\varpi)]}{2[1+d_{pb}(\omega,\varpi)]}\right)$$

By using (d_{pb2})

$$d_{pb}(\omega,\varpi)\leq \Upsilon_1 d_{pb}(\omega,\varpi)+\Upsilon_2 d_{pb}(\omega,\varpi)+\Upsilon_3 d_{pb}(\omega,\varpi)+\Upsilon_4 d_{pb}(\omega,\varpi)$$

$$(1-\Upsilon_1-\Upsilon_2-\Upsilon_3-\Upsilon_4)d_{pb}(\omega,\varpi)\leq 0$$

Therefor $d_{pb}(\omega,\varpi)\leq 0$. Thus $d_{pb}(\omega,\varpi)=0$. Then $\omega=\varpi$ contraction.

Corollary 3.5 Assume (Ψ,d_{pb}) a complete (P. B. M. S) with $b\geq 1$. Assume $T,F:\Psi\to\Psi$ self-map satisfying the following condition,

$$d_{pb}(T\tau,F\kappa)\leq \Upsilon d_{pb}(\tau,\kappa) \tag{3.7}$$

$\forall \tau,\kappa\in\Psi, 0<\Upsilon<1$. Then T,F has common unique (F. P).

Corollary 3.6 Assume (Ψ,d_{pb}) a complete (P. B. M. S) with $b\geq 1$. Assume $T,F:\Psi\to\Psi$ self-map satisfying the following condition,

$$d_{pb}(T\tau,F\kappa)\leq \Upsilon_1 d_{pb}(\tau,\kappa)+\Upsilon_2\left(\frac{[d_{pb}(\tau,T\tau)+d_{pb}(\kappa,F\kappa)][1+{}_{pb}(\tau,T\tau)]}{2[1+d_{pb}(\tau,\kappa)]}\right) \tag{3.8}$$

$\forall \tau,\kappa\in\Psi,, \Upsilon_1,\Upsilon_2\geq 0$ and $0<\Upsilon_1+\Upsilon_2<1$. Then T,F has common unique (F. P).

Corollary 3.7 Assume (Ψ, d_{pb}) a complete (P. B. M. S) with $b \geq 1$. Assume $T, F: \Psi \to \Psi$ self-map satisfying the following condition,

$$d_{pb}(T\tau, F\kappa) \leq \Upsilon_1 d_{pb}(\tau, \kappa) + \Upsilon_2 \left(\frac{[d_{pb}(\tau, T\tau) + d_{pb}(\kappa, F\kappa)][1 + {}_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) + \Upsilon_3 \left(\frac{[d_{pb}(\kappa, T\tau) + d_{pb}(\tau, F\kappa)][1 + {}_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) \tag{3.9}$$

$\forall \tau, \kappa \in \Psi, , \Upsilon_1, \Upsilon_2, \Upsilon_3 \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b < 1$. Then T, F has common unique (F. P).

Corollary 3.8 Assume (Ψ, d_{pb}) a complete (P. B. M. S) with $b \geq 1$. Assume $T, F: \Psi \to \Psi$ self-map satisfying the following condition,

$$d_{pb}(T\tau, F\kappa) \leq \Upsilon_1 d_{pb}(\tau, \kappa) + \Upsilon_2 \left(\frac{[d_{pb}(\tau, T\tau) + d_{pb}(\kappa, F\kappa)][1 + {}_{pb}(\tau, T\tau)]}{2[1 + d_{pb}(\tau, \kappa)]} \right) + \Upsilon_4 \left(\frac{[d_{pb}(\tau, \tau) + d_{pb}(\kappa, \kappa)][1 + d_{pb}(\tau, \kappa)]}{2[1 + {}_{pb}(\tau, T\tau)]} \right) \tag{3.10}$$

$\forall \tau, \kappa \in \Psi, , \Upsilon_1, \Upsilon_2, \Upsilon_3 \geq 0$ and $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 < 1$. Then T, F has common unique (F. P).

Example 3.1 Let $\Psi = [-10,10]$. Define the function $d_{pb}: \Psi \times \Psi \to [0, \infty)$ by $d_{pb}(\tau, \kappa) = (|\tau| + |\kappa| + 2)^2$ $\forall \tau, \kappa \in \Psi$. Then (Ψ, d_{pb}) is (P. B. M. S). Define $T: \Psi \to \Psi$ s.t $Tr = \frac{\tau}{10}$ $\forall \tau \in \Psi$. By applying theorem (3.1) with $\Upsilon_1 = 0.5$, $\Upsilon_2 = 0.4$, $\Upsilon_3 = \Upsilon_4 = o$, $b = 2$ we can show that show that T has (F. P) or not

$\Upsilon_1 + \Upsilon_2 + \Upsilon_3 b + \Upsilon_4 = (0.5) + (0.4) + 0 + 0 = 0.9$ so $0 < \Upsilon_1 + \Upsilon_2 + \Upsilon_3 b + \Upsilon_4 < 1$

$d_{pb}(\tau, \kappa) = (|\tau| + |\kappa| + 2)^2$

$$d_{pb}(T\tau, T\kappa) = d_{pb}\left(\frac{\tau}{10}, \frac{\kappa}{10}\right) = \left(\frac{|\tau|}{10} + \frac{|\kappa|}{10} + 2\right)^2 = \left(\frac{|\tau| + |\kappa| + 20}{10}\right)^2$$

$$d_{pb}(T\tau, \tau) = d_{pb}\left(\frac{\tau}{10}, r\right) = \left(\frac{|\tau|}{10} + |\tau| + 2\right)^2 = \left(\frac{11|\tau| + 20}{10}\right)^2$$

$$d_{pb}(T\kappa, \kappa) = d_{pb}\left(\frac{\kappa}{10}, \kappa\right) = \left(\frac{|\kappa|}{10} + |\kappa| + 2\right)^2 = \left(\frac{11|\kappa| + 20}{10}\right)^2$$

$$d_{pb}(T\tau, \kappa) = d_{pb}\left(\frac{\tau}{10}, \kappa\right) = \left(\frac{|\tau|}{10} + |\kappa| + 2\right)^2 = \left(\frac{|\tau| + 10|\kappa| + 20}{10}\right)^2$$

$$d_{pb}(T\kappa,\tau) = d_{pb}\left(\frac{\kappa}{10},\tau\right) = \left(\frac{|\kappa|}{10} + |\tau| + 2\right)^2 = \left(\frac{|\kappa| + 10|\tau| + 20}{10}\right)^2$$

Now, we can show that $\forall \tau,\kappa \in [-10,10]$ with $\Upsilon_1 = 0.5$, $\Upsilon_2 = 0.4$, $\Upsilon_3 = \Upsilon_4 = o$. Then

$$\left(\frac{|\tau| + |\kappa| + 20}{10}\right)^2 \le (0.5)(|\tau| + |\kappa| + 2)^2 + (0.4)\left[\frac{\left(\left(\frac{11|\tau| + 20}{10}\right)^2 + \left(\frac{11|\kappa| + 20}{10}\right)^2\right)\left(1 + \left(\frac{11|\tau| + 20}{10}\right)^2\right)}{2(1 + (|\tau| + |\kappa| + 2)^2)}\right]$$

Thus each condition of theorem (3.1) verified while 0 be unique (F. P) of T.

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