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A New Subclass of Meromorphic Univalent Functions defined by q-Salagean operator

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Abstract

In the present paper, we introduce a new subclass of meromorphic univalent functions, which are defined by quantum calculus. We define the class $\sum(n, q, \lambda, \alpha)$ of meromorphic univalent functions and obtain coefficient estimates, distortion bounds, closure theorems, Hadamard products, radii of starlikeness, convexity and close to convexity for function in this class. Further we study a class preserving integral operator. Finally, we determine partial sums for the functions of this class.

Keywords: Univalent function; Meromorphic function; q-derivative; Distortion bounds; Hadamard products; Partial sums.

AMS 2020 Mathematics subject classification 30C45.

1. Introduction

The family of meromorphic univalent functions $f(z)$ of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty} a_k z^k \tag{1}$$

in the open unit punctured disk $U^*=\{z: z\in C, 0<|z|<1\}$ is denoted by $\sum$. It is easy to verify that this function has a simple pole at the origin with residue 1.

Suppose that $\sum^*(\alpha)$ represent the subclass of meromorphic starlike functions of order α consisting of functions of the form (1) satisfying the criteria

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, (0\leq\alpha\leq 1) .$$

Similarly, a function $f(z)$ of the form (1) is said to be meromorphic convex functions of order α if it satisfy the following condition

$$-Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha\ , (0<\alpha<1) .$$

The class of meromorphic convex functions of order α is denoted by $\sum_c(\alpha)$. These classes were initially studied by Cluine[6] and followed by many researchers, see ([1], [2], [3], [7], [11], [12], [13]).

Let $\sum_v$ be the subclass of $\sum$ consisting of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}|a_k|\, z^k \quad . \tag{2}$$

The theory of q-calculus ($quantum\ calculus$) was first introduced by Jackson ([8], [9]). The application of q-calculus is a recent topic of study in univalent functions and it has sparked significant curiosity among scholars owing to its vast range of utilizations across diverse areas of Mathematics and theoretical Physics.

The basic concepts of q-calculus was found in the text book Aral et al [5] and a survey article by Srivastava et al ([10], [17]) Note worthy recent contribution in this area were found in ([4], [15], [16], [18]).

Tang et al. [18] defined the q-derivative for functions $f(z)$ of the form (1), as follows

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}$$

$$= -\frac{1}{qz^2} + \sum_{k=1}^{\infty} [k]_q\, a_k z^{k-1} \quad , \quad (0 < q < 1) \quad ,$$

(3)

where $[k]_q = \frac{1-q^k}{1-q}$.

It is easy to verify that $q \to 1^-$ $[1]_q \to 1$ and $lim_{q\to 1} D_q f(z) = f'(z)$.

Now, we define q-Salagean derivative for function $f(z)$ of the form (1) as

$$D_q^n f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} [k]_q^n\, a_k z^k \quad .$$

(4)

Using the above definition, we introduce a new subclass $\sum(n, q, \lambda, \alpha)$ of $\sum$ consisting of functions of $f(z)$ of the form (1) .

Definition 1.1: A function $f \in \sum$ is in the class $\sum(n,q,\lambda,\alpha)$ if it satisfies

$$-\operatorname{Re}\left\{\frac{(1+2\lambda)z\left(D_q^n f(z)\right)' + \lambda z^2\left(D_q^n f(z)\right)''}{(1+\lambda)D_q^n f(z)+\lambda z\left(D_q^n f(z)\right)'} - \alpha\right\} \geq 0 \qquad , \qquad (5)$$

for some $-1 \leq \lambda \leq 0$, $0 \leq \alpha \leq 1$, $n \in N_0$, $0 \leq q \leq 1$, where $D_n^q f(z)$ is a q-salagean derivative of f(z) defined by (4) .

$\sum_v^*(n,q,\lambda,\alpha) = \sum(n,q,\lambda,\alpha) \cap \sum_v$.

In the present article, we obtain coefficient bound, distortion bounds, convolution condition convex combination, radii of starlikeness, convexity and close-to-convexity. We also discuss a class preserving integral operator and partial sums for this class.

2. Main Results

In the first theorem of this section, we obtain sufficient condition for f(z) in the class $\sum(n,q,\lambda,\alpha)$.

Theorem 2.1: Let $f \in \sum$ If

$$\sum_{k=1}^{\infty}(k+\alpha)(1+\lambda+\lambda k)[k]_q^n\,|a_k| \leq (1-\alpha) \qquad (6)$$

then $f \in \sum(n,q,\lambda,\alpha)$,

where $z \in U^*$, $-1 \leq \lambda \leq 0$, $0 \leq \alpha \leq 1$, $n \in N_0$, $0 \leq q \leq 1$.

Proof: If the inequality (6) holds, then we have to show that

$$-\operatorname{Re}\left\{\frac{(1+2\lambda)z\left(D_q^n f(z)\right)' + \lambda z^2\left(D_q^n f(z)\right)''}{(1+\lambda)D_q^n f(z)+\lambda z\left(D_q^n f(z)\right)'} - \alpha\right\} \geq 0 \quad .$$

Let A(z) $= -(1+2\lambda)z\left(D_q^n f(z)\right)' - \lambda z^2\left(D_q^n f(z)\right)'' - \alpha\left\{(1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'\right\}$

B(z) $= (1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'$.

Now, we have to show that

$|A(z)+B(z)| - |A(z)-B(z)| > 0$.

Now, it follows that

$$|A(z)+B(z)|$$

$$= \left|-(1+2\lambda)z\left(D_q^n f(z)\right)' - \lambda z^2\left(D_q^n f(z)\right)'' - \alpha\left\{(1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'\right\} + (1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'\right|$$

$$= \left|\frac{2-\alpha}{z} - \sum_{k=1}^{\infty}[(1+2\lambda)k + \lambda k(k-1) + \alpha(1+\lambda) + \lambda\alpha k - (1+\lambda) - \lambda k][k]_q^n\, a_k\, z^k\right|$$

$$\geq \frac{2-\alpha}{|z|} - \sum_{k=1}^{\infty}[(1+2\lambda)k + \lambda k(k-1) + \alpha(1+\lambda) + \lambda\alpha k - (1+\lambda) - \lambda k]\,[k]_q^n\, |a_k|\, |z|^k$$

and

$|A(z)-B(z)|$

$$\Rightarrow \left|-(1+2\lambda)z\left(D_q^n f(z)\right)' - \lambda z^2\left(D_q^n f(z)\right)'' - \alpha\left\{(1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'\right\} - (1+\lambda)D_q^n f(z) + \lambda z\left(D_q^n f(z)\right)'\right|$$

$\Rightarrow \quad \left|\frac{-\alpha}{z} + \sum_{k=1}^{\infty}[-(1+2\lambda)k - \lambda k(k-1) - (1+\lambda)\alpha - \lambda\alpha - (1+\lambda) - \lambda k]\, [k]_q^n\, a_k\, z^k\right|$

$\leq \frac{\alpha}{|z|} \quad + \sum_{k=1}^{\infty\infty\infty}[-(1+2\lambda)k - \lambda k(k-1) - (1+\lambda)\alpha - \lambda\alpha - (1+\lambda) - \lambda k]\, [k]_q^n\, |a_k|\, |z|^k$.

Therefore, we conclude

$|A(z)+B(z)| - |A(z)-B(z)|$

$\geq \frac{2(1-\alpha)}{|z|} - \sum_{k=1}^{\infty} 2\big((1+2\lambda)k + \lambda k(k-1) + (1+\lambda)\alpha + \lambda\alpha k\big)\, [k]_q^n\, |a_k|\, |z|^k$

$\geq 2\left(\frac{1-\alpha}{|z|} - \sum_{k=1}^{\infty}(k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n\, |a_k|\, |z|^k\right)$.

The last expression is bounded below by 0 if

$$2\left((1-\alpha) - \sum_{k=1}^{\infty}(k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n\, |a_k|\right) \geq 0$$

$\Rightarrow \sum_{k=1}^{\infty}(k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n\, |a_k| \;\leq\; (1-\alpha)$.

and hence the proof is complete.

The result is sharp for the function

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1-\alpha}{(k+\alpha)(1+\lambda+\lambda k)[k]_q^n}\, x^k z^k, \quad (7)$$

where $\sum_{k=1}^{\infty}|x_k| = 1$.

Theorem 2.2: Let f $\in \sum_v$, then let f $\in \sum_v^*(n,q,\lambda,\alpha)$, then

$\sum_{k=1}^{\infty}(k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n\,|a_k| \leq (1-\alpha)$.

(8)

Proof: In view of Theorem 2.1, we need to prove that if f ∈ $\sum_v^*(n,q,\lambda,\alpha)$, then

$$-\mathrm{Re}\left\{\frac{(1+2\lambda)z\left(D_q^n f(z)\right)' + \lambda z^2\left(D_q^n f(z)\right)''}{(1+\lambda)D_q^n f(z)+\lambda z\left(D_q^n f(z)\right)'} - \alpha\right\} \geq 0 .$$

$$\mathrm{Re}\left\{\frac{-(1+2\lambda)z\left(D_q^n f(z)\right)' - \lambda z^2\left(D_q^n f(z)\right)'' - \alpha\left\{(1+\lambda)D_q^n f(z)+\lambda z\left(D_q^n f(z)\right)'\right\}}{\left((1+\lambda)D_q^n f(z)+\lambda z\left(D_q^n f(z)\right)'\right)}\right\} \geq 0 .$$

$$\mathrm{Re}\left\{\frac{-(1+2\lambda)z\left\{\frac{-1}{z^2}+\sum_{k=1}^{\infty}[k]_q^n\,a_k k z^{k-1}\right\}-\lambda z^2\left\{\frac{2}{z^3}+\sum_{k=1}^{\infty}[k]_q^n a_k k(k-1)z^{k-2}\right\}-\alpha\left\{(1+\lambda)\left\{\frac{1}{z}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k\right\}+\lambda z\left\{\frac{-1}{z^2}+\sum_{k=1}^{\infty}[k]_q^n a\right.\right.}{(1+\lambda)\left\{\frac{1}{z}+\sum_{k=1}^{\infty}[k]_q^n a_k z^k\right\}+\lambda z\left\{\frac{-1}{z^2}+\sum_{k=1}^{\infty}[k]_q^n a_k k z^{k-1}\right\}}\right.$$

≥ 0

$$\frac{1-\alpha}{z} - \sum_{k=1}^{\infty}(k+\lambda k+k^2\lambda+\alpha+\lambda\alpha+\lambda\alpha k)\,a_k[k]_q^n z^k \geq 0$$

$$\frac{1-\alpha}{z} - \sum_{k=1}^{\infty} k(1+\lambda+\lambda k)+\alpha(1+\lambda+\lambda k) \geq 0$$

$$\sum_{k=1}^{\infty}(k+\alpha)(1+\lambda+\lambda k) \;\leq\; (1-\alpha) .$$

Taking values of z on the real axis so that letting z → 1 , obtain the desired result.

Corollary 2.2: For f(z) ∈ $\sum_v^*(n,q,\lambda,\alpha)$,

$$a_k \leq \frac{1-\alpha}{\left[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n\right]},\ k \geq 1.$$

3. *Growth and Distortion theorems*

Theorem 3.1: For f ∈ $\sum_v^*(n,q,\lambda,\alpha)$ and $0 < |z| = r < 1$,

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\, r$$

and

$$|f(z)| \geq \frac{1}{r} - \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\, r \; .$$

The sharpness is attained for

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\, z \quad \text{at} \quad z = r, ir. \quad (9)$$

Proof: Since for k ≥ 1,

$$(1+2\lambda)(1+\alpha)\sum_{k=1}^{\infty}|a_k| \leq \sum_{k=1}^{\infty}(1+\lambda+\lambda k)\,(k+\alpha)\,[k]_q^n\,|a_k| \leq (1-\alpha)\,.$$

$$= \sum_{k=1}^{\infty}|a_k| \leq \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}$$

Since $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k\, z^k$.

From (4) and (9), we have

$$|f(z)| = \left|\frac{1}{z} + \sum_{k=1}^{\infty} a_k\, z^k\right|$$

$$\leq \frac{1}{|z|} + \sum_{k=1}^{\infty}|a_k|\,|z|^k$$

$$\leq \frac{1}{r} + r\sum_{k=1}^{\infty} a_k$$

$$\leq \frac{1}{r} + \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\, r\,.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - r\sum_{k=1}^{\infty}|a_k|$$

$$|f(z)| \geq \frac{1}{r} - \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\, r\,.$$

Theorem 3.2: Let f $\in \sum_v^*(n,q,\lambda,\alpha)$ then for $0 < \ |z| = r < 1$,

$$\left|D_q f(z)\right| \geq \frac{1}{qr^2} - \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}$$

and

$$\left|D_q f(z)\right| \leq \frac{1}{qr^2} + \frac{(1-\alpha)}{(1+2\lambda)(1+\alpha)}\ .$$

The results are sharp for the function defined by (9) at z= r, ir .

Proof: Note that f $\in \sum_v^*(n,q,\lambda,\alpha) \Leftrightarrow D_q f \in \sum_v^*(n,q,\lambda,\alpha)$. Using Theorem 2.1, we have

$$(1+2\lambda)(1+\alpha)\sum_{k=1}^{\infty}[k]_q^n\,|a_k| \leq \sum_{k=1}^{\infty}(1+\lambda+\lambda k)\,(k+\alpha)\,[k]_q^n\,|a_k| \leq (1-\alpha)$$

that is,

$$\sum_{k=1}^{\infty}[k]_q^n\,a_k \ \leq \frac{(1-\alpha)}{(1+\alpha)(1+2\lambda)} \qquad .$$

(10)

It follows from (5) and that(10)

$$\left|D_q f(z)\right| \geq \frac{1}{qr^2} - \sum_{k=1}^{\infty}[k]_q\ |a_k|$$

$$\geq \frac{1}{qr^2} - \frac{(1-\alpha)}{(1+\alpha)(1+2\lambda)}$$

and

$$\left|D_q f(z)\right| \leq \frac{1}{qr^2} + \sum_{k=1}^{\infty} [k]_q \; a_k$$

$$\leq \frac{1}{qr^2} + \frac{(1-\alpha)}{(1+\alpha)(1+2\lambda)} .$$

This completes the proof.

4. *Closure theorems*

Let f_j be defined, for j = 1, 2, ……… m, by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j}\, z^k \qquad (a_{k,j} \geq 0) . \tag{11}$$

Theorem 4.1: Let $f_j \in \sum_{v}^{*}(n, q, \lambda, \alpha)$, j = 1,2, ….m then

$$\mathrm{h}(z) = \sum_{j=1}^{m} c_j\, f_j(z) \left(c_j \geq 0, \sum_{j=1}^{m} c_j = 1\right) , \tag{12}$$

also, h(z) ∈ $\sum_{q,v}^{*}(n, q, \lambda, \alpha)$.

Proof: Using equation (11), we can write

$$\mathrm{h}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\sum_{J=1}^{m} c_j\, a_{k,j}\right) z^k .$$

Since $f_j \in \sum_{q,v}^{*}(n, q, \lambda, \alpha)$, then

$$\sum_{k=1}^{\infty} (k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n \left|a_{k,j}\right| \leq (1-\alpha) .$$

Now

$$\sum_{k=1}^{\infty} (k+\alpha)\,(1+\lambda+\lambda k)[k]_q^n \left(\sum_{j=1}^{m} c_j\, a_{k,j}\right)$$

$= \sum_{j=1}^{m} c_j \left[\sum_{k=1}^{\infty}(k+\alpha)(1+\lambda+\lambda k)[k]_q^n \, a_{k,j}\right]$

$\leq \left(\sum_{j=1}^{m} c_j\right)(1-\alpha) = (1-\alpha)$.

Hence $h(z) \in \sum_{q,v}^{*}(n,q,\lambda,\alpha)$.

Thus, the proof of Theorem 4.1 is complete.

Theorem 4.2: Let $f_0(z) = \frac{1}{z}$ and

$$f_k(z) = \frac{1}{z} + \frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)\,[k]_q^n]} \times z^k \qquad (k \geq 1) \qquad .$$

(13)

Then $f \in \sum_{q,v}^{*}(n,q,\lambda,\alpha) \Rightarrow f(z) = \sum_{k=0}^{\infty} \mu_k \, f_k(z)$,

where $\mu_k \geq 0 \ (k \geq 0)$, $\sum_{k=0}^{\infty} \mu_k = 1 \ and \ f_k(z)$, $k \geq 1$ are the extreme points.

Proof: Let

$$f(z) = \sum_{k=0}^{\infty} \mu_k \, f_k(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]} \times \mu_k z^k .$$

Then

$$\sum_{k=1}^{\infty} \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} \times \frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]} \times \mu_k$$

$= \sum_{k=1}^{\infty} \mu_k = 1 - \mu_0 \leq 1$.

So by Theorem 2.2, $f \in \sum_{v}^{*}(n,q,\lambda,\alpha)$.

Conversely, let $f \in \sum_{v}^{*}(n,q,\lambda,\alpha)$. Then

$$|a_k| \leq \frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]} \quad (k \geq 1).$$

By setting

$$\mu_k = \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} |a_k| \ (k \geq 1),$$

and

$$\mu_0 = 1 - \sum_{k=1}^{\infty} \mu_k \ ,$$

It can be seen that f can be written as in (13).

5. *Radii problem for the class* $\sum_v^*(\boldsymbol{n}, \boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{\alpha})$

In our next theorem, we obtain the radii of close-to- convexity starlikeness and convexity.

Theorem 5.1: Suppose f $\in \sum_{\mathrm{v}}^*(n, q, \lambda, \alpha)$. Then for $0 \leq \delta \leq 1$, f is

(i) close-to-convex of order δ in $|z| < r_1$,

$$r_1 = r_1(n, q, \lambda, \alpha, \delta)$$

$$:= \inf \left[\frac{(1-\delta)[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{k(1-\alpha)}\right]^{\frac{1}{(k+1)}} \tag{14}$$

(ii) starlike of order δ in $|z| < r_2$,

$$r_2 = r_2(n, q, \lambda, \alpha, \delta)$$

$$:= \inf \left[\frac{(1-\delta)[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)(k+2-\delta)}\right]^{\frac{1}{k+1}} \tag{15}$$

(iii) convex of order δ in $|z| < r_3$,

$$r_3 = r_3(n, q, \lambda, \alpha, \delta)$$

$$: \quad = \quad \inf \quad \left[\left[\frac{(1-\delta)\left[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n\right]}{(1-\alpha)(k+2-\delta)}\right]\right]^{\frac{1}{k+1}} ,$$

(16)

f(z) given by (16) gives the sharpness.

Proof: To prove (i) it may be shown that

$|z^2 f'(z) + 1| \leq 1 - \delta$ for $|z| < r_1(n, q, \lambda, \alpha, \delta)$.

From (4), we have

$$|z^2 f'(z) + 1| \leq \sum_{k=1}^{\infty} k|a_k|\,|z|^{k+1} .$$

Thus

$$|z^2 f'(z) + 1| \leq 1-\delta ,$$

If

$$\sum_{k=1}^{\infty} \left(\frac{k}{1-\delta}\right) |a_k||z|^{k+1} \leq 1 \quad .$$

(17)

But, by Theorem 2.2 and (17) will be true if

$$\left(\frac{k}{1-\delta}\right) |z|^{k+1} \leq \frac{(k+\alpha)(1+\lambda+\lambda k)[k]_q^n}{(1-\alpha)} ,$$

That is, if

$$|z| \leq \left[\frac{(1-\delta)\left[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n\right]}{k(1-\alpha)}\right]^{\frac{1}{k+1}} \quad (k \geq 1) ,$$

which gives (14).

To prove (ii) and (iii) it may be to show

$$\left|\frac{zf'(z)}{f(z)}+1\right| \leq 1-\delta \qquad \text{for } |z| < r_2, \tag{18}$$

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \leq 1-\delta \qquad \text{for } |z| < r_3, \tag{19}$$

respectively, by using arguments as in proving (i), we have the results. •

6. Hadamard products

To prove our results in this section, we will use the technique of Schild and Silverman [14].

Let f_j be defined by (11). Then for m = 2, the modified Hadamard product is defined by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1}\, a_{k,2}\, z^k = (f_2 * f_1). \tag{20}$$

Theorem 6.1: If $f_j \in \sum_{\mathrm{v}}^{*}(n, q, \lambda, \alpha)$, then

$$(f_1 * f_2)(z) \in \textstyle\sum_{\mathrm{v}}^{*}(n, \gamma, \lambda, \alpha),$$

where

$$\gamma = 1 - \frac{(k+1)\,(1-\alpha)^2}{(k+\alpha)^2(1+\lambda+\lambda k)[k]_q^n + (1-\alpha)^2} \ . \tag{21}$$

The result is sharp.

Proof: We need to find the largest γ such that

$$\sum_{k=1}^{\infty} \frac{[(k+\gamma)(1+\lambda+\lambda k)[k]_q^n]}{(1-\gamma)} \left|a_{k,1} a_{k,2}\right| \leq 1 \ .$$

Since

$$\sum_{k=1}^{\infty} \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} \qquad |a_{k,j}| \leq 1, (j=1,2) \qquad .$$

(22)

Then Cauchy-Schwarz inequality yields

$$\sum_{k=1}^{\infty} \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} \qquad \sqrt{|a_{k,1}a_{k,2}|} \leq 1 \qquad .$$

(23)

Thus it is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{[(k+\gamma)(1+\lambda+\lambda k)[k]_q^n]}{(1-\gamma)} \qquad |a_{k,1}a_{k,2}| \leq \sum_{k=1}^{\infty} \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} \qquad \sqrt{|a_{k,1}a_{k,2}|} \quad .$$

(24)

That is,

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n](1-\gamma)}{[(k+\gamma)(1+\lambda+\lambda k)[k]_q^n](1-\alpha)} \quad (k \geq 1).$$

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]} \quad (k \geq 1).$$

We need to prove that

$$\frac{(1-\alpha)}{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]} \leq \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n](1-\gamma)}{[(k+\gamma)(1+\lambda+\lambda k)[k]_q^n](1-\alpha)} \quad (k \geq 1),$$

Or, equivalently, that

$$\gamma = 1 - \frac{(k+1)\,(1-\alpha)^2}{(k+\alpha)^2(1+\lambda+\lambda k)[k]_q^n + (1-\alpha)^2} \quad (k \geq 1) \quad .$$

Since

$$\varphi_q(k) = 1 - \frac{(k+1)\,(1-\alpha)^2}{(k+\alpha)^2(1+\lambda+\lambda k)[k]_q^n + (1-\alpha)^2},$$

is an increasing function of k $(k \geq 1)$, letting k =1, we obtain

$$\gamma \leq \varphi_q(1) = 1 - \frac{2\,(1-\alpha)^2}{(2+\alpha)^2(1+2\lambda) + (1-\alpha)^2},$$

which is (19).

The result is sharp for the function $f_1(z) = f_2(z) = f(z)$ given by (9).

7. A family of integral operators

In this section, a family of integral operators for functions belonging to the class $\sum_{\mathrm{v}}^{*}(\mathrm{n}, \mathrm{q}, \lambda, \alpha)$ **.**

Theorem 7.1: Let $\mathrm{f}(z) \in \sum_{\mathrm{v}}^{*}(n, q, \lambda, \alpha)$. Then $F_c(z)$ defined by

$$F_c(z) = c\int_0^1 u^c\, f(uz)du \qquad (c > 0), \tag{25}$$

is in $\sum_{\mathrm{q,v}}^{*}(\delta)$, with

$$\delta = \delta(\alpha, \lambda, c) = 1 - \frac{2(1-\alpha)c}{(1-\alpha)c + (1+\alpha)(c+2)(1+2\lambda)}.$$

The result is best possible for $\mathrm{f}(z)$ defined by (9).

Proof: Suppose $\mathrm{f}(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in \sum_{\mathrm{v}}^{*}(n, q, \lambda, \alpha)$, we have

$$F_c(z) = c\int_0^1 u^c f(uz) \quad \mathrm{du} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{c}{k+c+1}\, a_k z^k. \tag{26}$$

It is sufficient to show that

$$\sum_{k=1}^{\infty} \frac{(k+\delta)c}{(1-\delta)(k+c+1)} |a_k| z^k \leq 1.$$

(27)

Since f(z) $\in \sum_{\mathrm{v}}^{*}(n, q, \lambda, \alpha)$, we have

$$\sum_{k=1}^{\infty} \frac{(k+\alpha)(1+\lambda+\lambda k)[k]_q^n}{(1-\alpha)} a_k \leq 1.$$

Thus (27) will be satisfied if

$$\frac{(k+\delta)c}{(1-\delta)(k+c+1)} \leq \frac{[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n]}{(1-\alpha)} .$$

For each k,

Or

$$\delta \leq 1 - \frac{(1-\alpha)(k+1)c}{(1-\alpha)c+ (k+c+1)(k+\alpha)(1+\lambda+\lambda k)[k]_q^n} .$$

(28)

Since the right side of (28) is an increasing function of k, by putting $k = 1$ in (23),

$$\delta \leq 1 - \frac{2(1-\alpha)c}{(1-\alpha)c + (c+2)(1+\alpha)(1+2\lambda)} .$$

Hence the theorem is proved. ●

8. Partial sums

Let f(z) of the form (1),

$$f_s(z) = \frac{1}{z} + \sum_{k=1}^{s} a_k z^k .$$

and

$$\varphi_{q,k}(\alpha,\lambda) = \left[(k+\alpha)(1+\lambda+\lambda k)[k]_q^n\right] . \quad (29)$$

In the next theorem, we determine bounds for $\frac{f(z)}{f_s(z)}$.

Theorem 8.1: If f ∈ $\sum$, satisfies (12), then

$$\text{Re}\left(\frac{f(z)}{f_s(z)}\right) \geq \frac{\varphi_{q,s+1} - (1-\alpha)}{\varphi_{q,s+1}} ,$$

where

$$\varphi_{q,k} \geq \begin{cases} (1-\alpha), & if\ k = 1,2,\ldots\ldots.s \\ \varphi_{q,s+1}, & if\ k \geq s+1 \end{cases} . \quad (30)$$

The result is sharp for function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)}{\varphi_{q,s+1}} z^{s+1} . \quad (31)$$

Proof: Define w(z) by

$$\frac{1+w(z)}{1-w(z)} = \frac{\varphi_{q,s+1}}{(1-\alpha)}\left[\frac{f(z)}{f_s(z)} - \frac{\varphi_{q,s+1} - (1-\alpha)}{\varphi_{q,s+1}}\right] = \frac{1 + \sum_{k=1}^{s} a_k z^{k+1} + \left(\frac{\varphi_{q,s+1}}{(1-\alpha)}\right)\sum_{k=s+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{s} a_k z^{k+1}} . \quad (32)$$

It must be shown that $|w| \leq 1$. Now (32) yields

$$w(z) = \frac{\left(\frac{\varphi_{q,s+1}}{(1-\alpha)}\right)\sum_{k=s+1}^{\infty} a_k z^{k+1}}{2+2\sum_{k=1}^{s} a_k z^{k+1} + \frac{\varphi_{q,s+2}}{(1-\alpha)}\sum_{k=s+1}^{\infty} a_k z^{k+1}} .$$

Hence

$$|w(z)| \le \frac{\left(\frac{\varphi_{q,s+1}}{(1-\alpha)}\right)\sum_{k=s+1}^{\infty}|a_k|}{2-2\sum_{k=1}^{s}|a_k| - \left(\frac{\varphi_{q,s+1}}{(1-\alpha)}\right)\sum_{k=s+1}^{\infty}|a_k|}.$$

Now , $|w| \le 1$ if and only if

$$\left(\frac{\varphi_{q,s+1}}{(1-\alpha)}\right)\sum_{k=s+1}^{\infty}|a_k| \le 2 - 2\sum_{k=1}^{s}|a_k| ,$$

or, equivalently,

$$\sum_{k=1}^{s}|a_k| + \sum_{k=s+1}^{\infty}\frac{\varphi_{q,s+1}}{(1-\alpha)}|a_k| \le 1.$$

From (12), it is sufficient to show that

$$\sum_{k=1}^{s}|a_k| + \sum_{k=s+1}^{\infty}\frac{\varphi_{q,s+1}}{(1-\alpha)}|a_k| \le \sum_{k=1}^{\infty}\frac{\varphi_{q,k}}{(1-\alpha)}|a_k| ,$$

That is

$$\sum_{k=1}^{s}\left(\frac{\varphi_{q,k}-(1-\alpha)}{(1-\alpha)}\right)|a_k| + \sum_{k=s+1}^{\infty}\left(\frac{\varphi_{q,k}-\varphi_{q,s+1}}{(1-\alpha)}\right)\times|a_k| \ge 0. \quad (33)$$

For $z = re^{\frac{i\pi}{s+2}}$ we have

$$\frac{f(z)}{f_s(z)} = 1 + \frac{(1-\alpha)}{\varphi_{q,s+1}}z^{s+2} \to 1 - \frac{(1-\alpha)}{\varphi_{q,s+1}}$$

$$= \frac{\varphi_{q,s+1}-(1-\alpha)}{\varphi_{q,s+1}} \text{ where } r \to 1^- ,$$

which shows that f (z) given by (31) gives the sharpness. ●

In the next theorem, we determine bounds for $\frac{f_s(z)}{f(z)}$.

Theorem 8.2: If f ∈ $\sum$, satisfies (12), then

$$\mathrm{Re}\left(\frac{f_s(z)}{f(z)}\right) \geq \frac{\varphi_{q,s+1}}{\varphi_{q,s+1}+(1-\alpha)}, \quad (34)$$

where $\varphi_{q,s+1}$ defined by (29) and satisfies (30) and the result is sharp for the function defined by (31).

Proof: The proof follows by defining

$$\frac{1+w}{1-w} = \frac{\varphi_{q,s+1}+(1-\alpha)}{(1-\alpha)} \times \left[\frac{f_s(z)}{f(z)} - \frac{\varphi_{q,s+1}}{\varphi_{q,s+1}+(1-\alpha)}\right].$$

The reminder part is as in theorem 8.1 so, we exclude it here.

In theorem 8.3, we determine bounds for $\frac{D_q f(z)}{D_q f_s(z)}$ and $\frac{D_q f_s(z)}{D_q f(z)}$.

Theorem 8.3: If f ∈ $\sum$, satisfies (12), then

$$\mathrm{Re}\left(\frac{D_q f(z)}{D_q f_s(z)}\right) \geq \frac{\varphi_{q,s+1}-[s+1]_q(1-\alpha)}{\varphi_{q,s+1}} \quad (35)$$

and

$$\mathrm{Re}\left(\frac{D_q f_s(z)}{D_q f(z)}\right) \geq \frac{\varphi_{q,s+1}}{\varphi_{q,s+1}+[s+1]_q(1-\alpha)} \quad (36)$$

where $\varphi_{q,s+1} \geq [s+1]_q(1-\alpha)$ and

$$\varphi_{q,k} \geq \begin{cases} [k]_q(1-\alpha) & if\ k = 1,2,\dots\dots s \\ [k]_q\left(\frac{\varphi_{q,s+1}}{[s+1]_q}\right), & if\ k \geq s \end{cases}.$$

(37)

The result is sharp for the function defined by (31).

Proof: We write

$$\frac{1+w(z)}{1-w(z)} = \frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)} \times \left[\frac{D_q f(z)}{D_q f_s(z)} - \left(\frac{\varphi_{q,s+1} - [s+1]_q(1-\alpha)}{\varphi_{q,s+1}}\right)\right],$$

Where

$$\mathrm{w} = \frac{\left(\frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)}\right)\sum_{k=s+1}^{\infty}[k]_q\, a_k z^{k+1}}{2+2\sum_{k=1}^{s+1}[k]_q a_k z^{k+1} + \left(\frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)}\right)\sum_{k=s+1}^{\infty}[k]_q a_k z^{k+1}}.$$

Now, $|w| \leq 1$if and only if

$$\sum_{k=1}^{s}[k]_q|a_k| + \left(\frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)}\right)\sum_{k=s+1}^{\infty}[k]_q\,|a_k| \leq 1\,.$$

From (6), it is sufficient to show that

$$\sum_{k=1}^{s}[k]_q\,|a_k| + \left(\frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)}\right)\sum_{k=s+1}^{\infty}[k]_q\,|a_k| \leq \sum_{k=1}^{\infty}\frac{\varphi_{q,k}}{(1-\alpha)}\,|a_k|.$$

That is

$$\sum_{k=1}^{s}\left(\frac{\varphi_{q,k} - [k]_q(1-\alpha)}{(1-\alpha)}\right)|a_k| + \sum_{k=s+1}^{\infty}\left(\frac{[s+1]_q\,\varphi_{q,k} - [k]_q\,\varphi_{q,s+1}}{[s+1]_q(1-\alpha)}\right)|a_k| \geq 0.$$

To prove (36), define the function w by

$$\frac{1+w}{1-w} = \frac{[s+1]_q(1-\alpha)+\varphi_{q,s+1}}{[s+1]_q(1-\alpha)} \times \left[\frac{D_q f_s(z)}{D_q f(z)} - \frac{\varphi_{q,s+1}}{[s+1]_q(1-\alpha)+\varphi_{q,s+1}^{n}}\right],$$

and by similar arguments in the first part, we get the desired result.

Conclusion.

In this article, we introduced and studied a new subclass of meromorphic functions with positive coefficients by applying q-derivative operator. We obtained coefficient inequalities, distortion theorems, closure properties, radius results, and sufficient conditions for starlikeness, convexity, and close-to-convexity for functions belonging to the defined subclass. The results established in this work not only unify various existing results available in the literature but also extend many earlier investigations related to meromorphic starlike function classes. The inclusion of the q-derivative operator provides a broader framework and reveals new connections between q-calculus and geometric function theory.

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